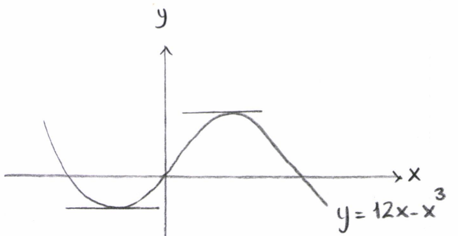
MATH 119 _ CALCULUS WITH ANALYTIC GEOMETRY

RECITATION 4

1. Find the points on the graph of $f(x) = 12x - x^3$ where the tangent line is horizontal.

Solution:



Tangent line is horizontal when the slope of the line is 0.

The slope of the tangent line to the graph of $y = f(x)$ at $(x_0, f(x_0))$ is given by $m = f'(x_0)$.

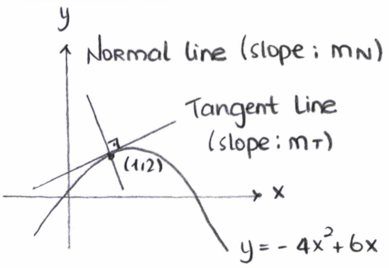
So find the points such that $f'(x_0) = 0$.

$f'(x) = 12 - 3x^2 = 0 \qquad 3(2-x)(2+x) = 0 \Rightarrow x = \pm 2$

Thus; at $(-2, f(-2))$ and $(2, f(2))$ the tangent line is horizontal.

2. Find the equations of the tangent and normal line at the point $(1,2)$ to the graph of $f(x) = -4x^2 + 6x$.

Solution:



Check that $(1,2)$ is on the $y = f(x)$

$f(1) = -4 + 6 = 2$ ✓

The slope of the tangent line:

$f'(x) = -8x + 6 \Rightarrow f'(1) = -8 + 6 = -2$

Thus; the equation of the tangent line is

$y - 2 = -2(x-1)$

The slope of the normal line:

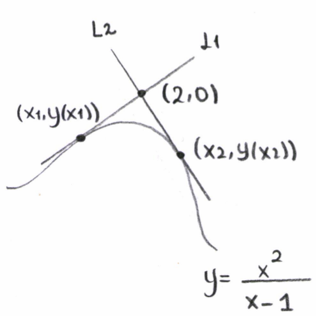
$m_T \cdot m_N = -1$ since $m_T \perp m_N$: $(-2) \cdot m_N = -1 \Rightarrow m_N = 1/2$

The equation of the normal line is ; $y - 2 = \frac{1}{2}(x-1)$

3. Find two straight lines that are tangent to the curve $y = \dfrac{x^2}{(x-1)}$ and pass through the point $(2,0)$.

Solution:

$y(2) = \dfrac{4}{2-1} = 4 \neq 0$, $(2,0)$ is not on the curve.



The slopes of L_1 and L_2 : $y' = \dfrac{2x(x-1) - x^2}{(x-1)^2} = \dfrac{2x^2 - 2x - x^2}{(x-1)^2} = \dfrac{x^2 - 2x}{(x-1)^2}$

$m_{L1} = \dfrac{x_1^2 - 2x_1}{(x_1-1)^2}$ and $m_{L2} = \dfrac{x_2^2 - 2x_2}{(x_2-1)^2}$

Consider L_1 passing through $(2,0)$ and $\left(x_1, \dfrac{x_1^2}{x_1-1}\right)$

So the slope is $m_{L1} = \dfrac{0 - \dfrac{x_1^2}{x_1-1}}{2 - x_1} = \dfrac{x_1^2}{(x_1-1)(x_1-2)}$

(Find x_1 and x_2)

$\dfrac{\cancel{x_1}(x_1-2)}{(x_1-1)^{\cancel{2}}} = \dfrac{x_1^{\cancel{2}}}{\cancel{(x_1-1)}(x_1-2)} \Rightarrow (x_1-2)^2 = x_1(x_1-1)$

$x_1^2 - 4x_1 + 4 = x_1^2 - x_1 \Rightarrow x_1 = \dfrac{4}{3}$

Thus ; the tangent line at $\left(\frac{4}{3}, y\left(\frac{4}{3}\right)\right)$ is passing through $(2,0)$:

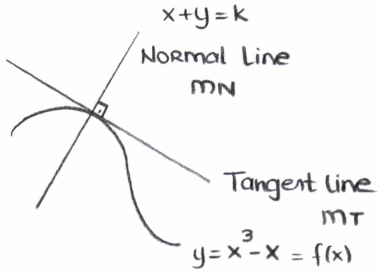
$$y'\left(\frac{4}{3}\right) = \frac{\left(\frac{4}{3}\right)^2 - 2 \cdot \frac{4}{3}}{\left(\frac{4}{3}-1\right)^2} = -8 \quad \text{and} \quad y\left(\frac{4}{3}\right) = \frac{\left(\frac{4}{3}\right)^2}{\frac{4}{3}-1} = \frac{16}{3}$$

The equation is $y - \frac{16}{3} = -8\left(x - \frac{4}{3}\right)$

(Do the same calculations for L2)

4. For what values of k , is the line $x+y=k$ normal to the curve $y=x^3-x$?

Solution:



The slope of $x+y=k \Rightarrow y=-x+k \quad m_N=-1$

And we have $m_N . m_T = -1 \Rightarrow (-1). m_T = -1 \Rightarrow m_T = 1$

Firstly, find the points such that $f'(x_0) = 1$.

$f'(x) = 3x^2 - 1 = 1 \Rightarrow 3x^2 = 2 \quad x = \pm \frac{\sqrt{2}}{\sqrt{3}}$

If $x+y=k$ is passing through $\left(\frac{\sqrt{2}}{\sqrt{3}}, f\left(\frac{\sqrt{2}}{\sqrt{3}}\right)\right)$ and $\left(-\frac{\sqrt{2}}{\sqrt{3}}, f\left(-\frac{\sqrt{2}}{\sqrt{3}}\right)\right)$ then it is normal to the curve $y=x^3-x = f(x)$

$f\left(\frac{\sqrt{2}}{\sqrt{3}}\right) = \frac{2\sqrt{2}}{3\sqrt{3}} - \frac{\sqrt{2}}{\sqrt{3}}$; $k = \frac{\sqrt{2}}{\sqrt{3}} + \frac{2\sqrt{2}}{3\sqrt{3}} - \frac{\sqrt{2}}{\sqrt{3}} = \frac{2\sqrt{2}}{3\sqrt{3}}$

$f\left(-\frac{\sqrt{2}}{\sqrt{3}}\right) = -\frac{2\sqrt{2}}{3\sqrt{3}} + \frac{\sqrt{2}}{\sqrt{3}}$; $k = -\frac{\sqrt{2}}{\sqrt{3}} - \frac{2\sqrt{2}}{3\sqrt{3}} + \frac{\sqrt{2}}{\sqrt{3}} = -\frac{2\sqrt{2}}{3\sqrt{3}}$

The Product Rule: If functions f and g are differentiable at x ;

$(f.g)'(x) = f'(x).g(x) + f(x).g'(x)$

The Quotient Rule: If f and g are differentiable at x and $g(x) \neq 0$

$$\left(\frac{f}{g}\right)'(x) = \frac{f'(x).g(x) - f(x).g'(x)}{[g(x)]^2}$$

5. Suppose that $f(2)=3$, $f'(2)=4$, $g(2)=5$ and $g'(2)=6$. Find the followings:

(a) $(f.g)'(2)$

(b) $(f^2.g)'(2)$

(c) $\left(\frac{f}{g^2}\right)'(2)$

Solution:

(a) $(f.g)'(2) = f'(2).g(2) + f(2).g'(2)$

$= 4.5 + 3.6 = 20 + 18 = 38$

(b) $(f^2.g)'(2) = 2f(2).f'(2).g(2) + f^2(2).g'(2)$

$= 2.3.4.5 + 3^2.6 = 120 + 54 = 174$

(c) $\left(\frac{f}{g^2}\right)'(2) = \frac{f'(2).g^2(2) - f(2).2g(2).g'(2)}{g^4(2)} = \frac{4.5^2 - 3.2.5.6}{5^4}$

The Chain Rule: If $f(u)$ is differentiable at $u=g(x)$ and g is differentiable at x

$(f \circ g)'(x) = f'(g(x)).g'(x)$.

6. Find $F'(x_0)$ by using the giving information;

(a) $F(x) = f(2f(4f(x)))$, $x_0=0$, $f(0)=0$ and $f'(0)=2$.

(b) $F(x) = f(x.f(x))$, $x_0=1$, $f(1)=2$, $f(2)=3$, $f'(1)=4$, $f'(2)=5$.

Solution:

(a) $F'(x) = f'(2f(4f(x))).2.f'(4f(x)).4.f'(x)$

$F'(0) = f'(2.f(4.f(0))).2.f'(4.f(0))\,4.f'(0)$

$= f'(2.f(0)).2.f'(0).4.2 = 16.f'(0).f'(0) = 16.2.2 = 64$

(b) $F'(x) = f'(x.f(x)).[f(x) + x.f'(x)]$

$F'(1) = f'(1.f(1))\,[f(1) + f'(1)]$

$= f'(2).[2 + f'(1)] = 5.(2+4) = 30$

7. Find the derivative dy/dx for each of the following functions:

(a) $y = \sqrt{x + \sqrt{1+x}}$

(b) $y = \left(\frac{2x+3}{4x+5}\right)^6$

(c) $y = \frac{x^2 + \sqrt[3]{x}}{x^3 + \sqrt{x}}$

(d) $y = \sqrt{x}\,(1 - x + x^2 - x^3)(1+x)$

Solution:

(a) $\frac{dy}{dx} = \frac{1}{2} \cdot \frac{1}{\sqrt{x + \sqrt{1+x}}} \quad \left(1 + \frac{1}{2} \cdot \frac{1}{\sqrt{1+x}}\right)$

(b) $\frac{dy}{dx} = 6 \cdot \left(\frac{2x+3}{4x+5}\right)^5 \cdot \left[\frac{2(4x+5) - 4(2x+3)}{(4x+5)^2}\right]$

(c) $\frac{dy}{dx} = \frac{\left(2x + \frac{1}{3} \cdot x^{-2/3}\right)(x^3 + \sqrt{x}) - (x^2 + \sqrt[3]{x})\left(3x^2 + \frac{1}{2\sqrt{x}}\right)}{(x^3 + \sqrt{x})^2}$

The Derivatives of sine & Cosine

$\frac{d}{dx} \sin x = \cos x$

$\frac{d}{dx} \cos x = -\sin x$

8. Find the derivative dy/dx for each of the following functions:

(a) $y = x^3 \cdot \cos(x^2 + x)$

(b) $y = \sin(\sin(\sin x))$

(c) $y = \sec^2(x^3) \cdot \sin(x^3)$

(d) $y = \tan^2(\sec(x^2))$

Solution:

(a) $\frac{dy}{dx} = 3x^2 \cdot \cos(x^2 + x) + x^3 \left(-\sin(x^2 + x) \cdot (2x+1)\right)$

(b) $\frac{dy}{dx} = \cos(\sin(\sin x)) \cdot \cos(\sin x) \cdot \cos x$

(c) $\frac{d}{dx} \sec x = \frac{d}{dx} \frac{1}{\cos x} = -\frac{1}{\cos^2 x} \cdot (-\sin x) = \tan x \cdot \sec x$

$\frac{dy}{dx} = 2 \cdot \sec(x^3) \cdot \tan(x^3) \cdot \sec(x^3) \cdot 3x^2 \cdot \sin(x^3) + \sec^2(x^3) \cdot \cos(x^3) \cdot 3x^2$

9. Find all points on the graph of the function $f(x) = \sin(2x) - 2\sin x$ at which the tangent line is horizontal.

Solution :

$f'(x) = \cos(2x).2 - 2\cos x = 0$

$\cos(2x) - \cos x = 0 \Rightarrow 2\cos^2 x - 1 - \cos x = 0 \Rightarrow 2t^2 - t - 1 = 0$

$\cos 2x = 2\cos^2 x - 1$

$$\begin{array}{cc} 2t & 1 \\ t & -1 \end{array}$$

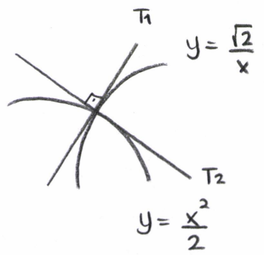
$(2\cos x + 1)(\cos x - 1) = 0 \Rightarrow \cos x = -\frac{1}{2}$ $\qquad \cos x = 1$

$x = \frac{2\pi}{3} + n.2\pi \qquad x = 0 + n.2\pi \quad ; \quad n \in \mathbb{Z}$

$x = -\frac{2\pi}{3} + n.2\pi$

10. Show that the cueves $xy = \sqrt{2}$ and $x^2 = 2y$ intersect at the point $P(\sqrt{2}, 1)$ and their tangent lines at P are peependicular to each other.

Solution:



$(\sqrt{2}, 1)$ satisfies the cueves $xy = \sqrt{2}$ and $x^2 = 2y$

$\sqrt{2}.1 = \sqrt{2}$ ✓ $\qquad 2 = 2.1$ ✓

So $(\sqrt{2}, 1)$ is the intersection point.

The slope of T_1: $y' = -\frac{\sqrt{2}}{x^2} \Rightarrow y'(\sqrt{2}) = -\frac{\sqrt{2}}{2} = m_1$

The slope of T_2: $y' = x \Rightarrow y'(\sqrt{2}) = \sqrt{2} = m_2$

Since $m_1.m_2 = -\frac{\sqrt{2}}{2}.\sqrt{2} = -1$, T_1 and T_2 are peependicular to each other.

11. Calculate enough deeivatives of the given function to enable you to guess the geneeal formula for $f^{(n)}(x)$.

(a) $f(x) = \frac{1}{3-x}$

(b) $f(x) = \sqrt{x}$

(c) $f(x) = x.\cos x$

Solution:

(a) $f(x) = \frac{1}{3-x}$

$f'(x) = -\frac{1}{(3-x)^2}$

$f''(x) = (-1).(-2)\ \frac{1}{(3-x)^3}$

$f'''(x) = (-1)(-2)(-3)\ \frac{1}{(3-x)^4}$

$f^{(4)}(x) = (-1)(-2)(-3)(-4)\ \frac{1}{(3-x)^5}$

$\vdots$

$f^{(n)} = (-1)^n . n!\ \frac{1}{(3-x)^{n+1}}$

(b) $f(x) = \sqrt{x} = x^{1/2}$

$f'(x) = \frac{1}{2}\ x^{-1/2}$

$f''(x) = (\frac{1}{2})(-\frac{1}{2})\ x^{-3/2}$

$f'''(x) = (\frac{1}{2})(-\frac{1}{2})(-\frac{3}{2})\ x^{-5/2}$

$f^{(4)}(x) = (\frac{1}{2})(-\frac{1}{2})(-\frac{3}{2})(-\frac{5}{2})\ x^{-7/2}$

$\vdots$

$f^{(n)}(x) = \frac{(-1)^{n-1}}{2^n}\ x^{-\frac{(2n-1)}{2}}$