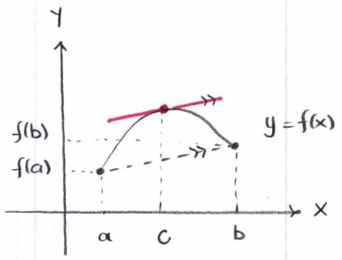
MATH 119 - CALCULUS WITH ANALYTIC GEOMETRY

RECITATION 5

The Mean-Value Theorem: Suppose that f is continuous on $[a,b]$ and differentiable on (a,b). Then there exists $c \in (a,b)$ such that

$$f'(c) = \frac{f(b)-f(a)}{b-a}$$



1. Let f be differentiable on $\mathbb{R}$. Suppose that $f'(x) \neq 0$ for any $x \in \mathbb{R}$. Prove that f has at most one real root.

Solution:

Assume that there exist $c_1, c_2 \in \mathbb{R}$, $c_1 \neq c_2$ and $f(c_1) = f(c_2) = 0$ (Assume $c_1 < c_2$)

Since f is differentiable on $\mathbb{R}$; f is continuous on $[c_1, c_2]$ and differentiable on (c_1, c_2)

By the Mean-Value Theorem, there exists $c \in (c_1, c_2)$ such that

$f'(c) = \frac{f(c_2)-f(c_1)}{c_2-c_1} = \frac{0-0}{c_2-c_1} = 0$ but that contradicts with $f'(x) \neq 0$ for any $x \in \mathbb{R}$.

Thus; the assumption is wrong. f has at most one real root.

2. Show that $x^3 + x^2 + 3x + 7 = 0$ has exactly one real root.

Solution:

Define $f(x) = x^3 + x^2 + 3x + 7$, f is continuous and differentiable for all $x \in \mathbb{R}$, since it is polynomial.

$\left.\begin{array}{l} f(-2) = -8+4-6+7 < 0 \\ f(0) = 7 > 0 \end{array}\right\}$ f is continuous on $[-2,0]$ and 0 is between $f(-2)$ & $f(0)$ by the Intermediate-Value Theorem there exists $c \in [-2,0]$ such that $f(c) = 0$.

c is a real root of $x^3 + x^2 + 3x + 7 = 0$.

CLAIM: c is unique.

Assume that there exists another root of the equation, $c' \in \mathbb{R}$, $f(c') = 0$ and $c \neq c'$.
Assume $c' > c$

f is continuous on $[c,c']$ and differentiable on (c,c'). $f(c) = f(c') = 0$ by the Rolle's Theorem there exists $m \in (c,c')$ such that $f'(m) = 0$.

$f'(x) = 3x^2 + 2x + 3$ $\Delta = 4 - 4.3.3 < 0$, $f'(x)$ has no real root. We get a contradiction, the assumption is wrong.

Thus; c is unique.

The Intermediate-Value Theorem: If $f(x)$ is continuous on $[a,b]$ and s is between $f(a)$ and $f(b)$ then there exists $c \in [a,b]$ such that $f(c) = s$.

Rolle's Theorem: let f be continuous on $[a,b]$ and differentiable on (a,b). If $f(a) = f(b)$ then there exists $c \in (a,b)$ such that $f'(c) = 0$.

3. Show that $6x^4 - 7x + 1$ does not have more than two distinct real roots.

Solution:

Define $f(x) = 6x^4 - 7x + 1$, f is continuous and differentiable for all $x \in \mathbb{R}$.

$f(0) = 1 > 0$

$f(\frac{1}{2}) = \frac{6}{16} - \frac{7}{2} + 1 < 0$

$f(2) = 2.16 - 7.2 + 1 > 0$

f is continuous on $[0, \frac{1}{2}]$ and 0 between $f(0)$ & $f(\frac{1}{2})$ then there exists $c_1 \in [0, \frac{1}{2}]$ such that $f(c_1) = 0$ by the IVT.

f is continuous on $[\frac{1}{2}, 2]$ and 0 is between $f(\frac{1}{2})$ & $f(2)$ then there exists $c_2 \in [\frac{1}{2}, 2]$ such that $f(c_2) = 0$ by the IVT.

Assume that there exists $c_3 \in \mathbb{R}$ such that $c_1 \neq c_2 \neq c_3$ and $f(c_3) = 0$ (Assuming $c_1 < c_2 < c_3$)

f is continuous on $[c_1, c_2]$ and differentiable on (c_1, c_2). We have $f(c_1) = f(c_2)$ by Rolle's Theorem there exists $k_1 \in (c_1, c_2)$ such that $f'(k_1) = 0$.

Similarly, f is continuous on $[c_2, c_3]$ and differentiable on (c_2, c_3). We have $f(c_2) = f(c_3)$ by Rolle's Theorem there exists $k_2 \in (c_2, c_3)$ such that $f'(k_2) = 0$.

$f'(x) = 24x^3 - 7$ (Show that $f'(x)$ has only one real root)

4. By using the MVT, show that $\tan x > x$ for $0 < x < \frac{\pi}{2}$.

Solution:

Define $f(t) = \tan t - t$, f is continuous for all $t \in (0, \frac{\pi}{2})$.

Let $0 < x < \frac{\pi}{2}$, apply the MVT to f on the interval $[0, x]$.

f is continuous on $[0,x]$ and differentiable on $(0,x)$, by MVT there exists $c \in (0,x)$ such that

$$f'(c) = \frac{f(x) - f(0)}{x - 0} = \frac{\tan x - x - 0}{x} = \frac{\tan x - x}{x}$$

$$f'(c) = 1 + \tan^2 c - 1 = \tan^2 c > 0 \text{ for all } c \in (0,x)$$

Thus; $\frac{\tan x - x}{x} > 0$ since $x > 0$, $\tan x - x > 0 \Rightarrow \tan x > x$ for $0 < x < \frac{\pi}{2}$

5. Suppose that $f(x)$ is continuous on $[-7, 0]$, differentiable on $(-7, 0)$ and suppose that $f(-7) = -3$ and $f'(x) \leq 2$ for any $x \in (-7, 0)$. What is the largest possible value for $f(0)$?

Solution:

f is continuous on $[-7, 0]$ and differentiable on $(-7, 0)$. By the MVT, there exists $c \in (-7, 0)$ such that

$$f'(c) = \frac{f(0) - f(-7)}{0 - (-7)} = \frac{f(0) - (-3)}{7} = \frac{f(0) + 3}{7}$$

It is given that $f'(x) \leq 2$ so $f'(c) \leq 2 \Rightarrow \frac{f(0)+3}{7} \leq 2$

$f(0) + 3 \leq 14 \Rightarrow f(0) \leq 11$. Thus; the largest possible value for $f(0)$ is 11.

Theorem: Let J be an open interval and I be an interval consisting all points of J and possibly one or both of the endpoints of J. Suppose f is continuous on I, differentiable on J.

(i) If $f'(x) > 0$ for all $x \in J$, f is increasing on I.

(ii) If $f'(x) < 0$ for all $x \in J$, f is decreasing on I.

$$\left(\begin{array}{l} f'(x) \geq 0 \Rightarrow \text{non-decreasing} \\ f'(x) \leq 0 \Rightarrow \text{non-increasing} \end{array} \right)$$

6. Find the intervals of increase and decrease of the following functions.

(a) $f(x) = x^3 + 3x^2 - x + 1$.

(b) $f(x) = x + 2\cos x$

Solution:

(a) $f'(x) = 3x^2 + 6x - 1$

$\Delta = 36 - 4.3.(-1) = 36 + 12 = 48$

$x_1 = \frac{-6 + \sqrt{48}}{6} = -1 + \frac{2\sqrt{3}}{3}$ $\quad$ $x_2 = -1 - \frac{2\sqrt{3}}{3}$

x	$-\infty$ $\quad$ x_2	x_1	∞
f'	+	−	+
f	↗	↘	↗

Thus; f is increasing on $(-\infty, x_2)$ & (x_1, ∞)
decreasing on (x_2, x_1).

(b) $f'(x) = 1 - 2\sin x$

$1 - 2\sin x = 0 \Rightarrow \sin x = \frac{1}{2}$ $\quad$ $x = \frac{\pi}{6} + 2n\pi$ OR $x = \frac{5\pi}{6} + 2n\pi$ $\quad$ $n \in \mathbb{Z}$.

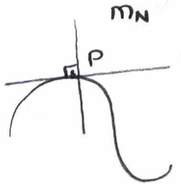
$f'(x) > 0$ when $1 - 2\sin x > 0 \Rightarrow \sin x < \frac{1}{2}$

Restrict the domain on $[0, 2\pi]$ $\quad$ on $[\frac{\pi}{6}, \frac{5\pi}{6}]$ f is decreasing for other points f is increasing.

Implicit Differentiation: Let $F(x,y) = 0$ be given implicitly. To find dy/dx by implicit differentiation, differentiate both sides of the equation with respect to x, regarding y as a function of x and using the chain rule to differentiate functions of y. And collect terms dy/dx on one side.

7. Find the equations of the tangent and normal lines drawn to the graph of $y=y(x)$ given implicitly by $x+\tan\frac{y}{x}=2$ at the point $P=(1,\frac{\pi}{4})$.

Solution:



P is on the graph:

$1+\tan\left(\frac{\pi}{4}\right)=2 \Rightarrow 2=2$ ✓

We have $m_T=y'(1)$.

Differentiate $x+\tan\frac{y}{x}=2$ by implicit differentiation assuming $y=y(x)$

$1+\left[1+\tan^2\left(\frac{y}{x}\right)\right]\cdot\frac{y'.x+y}{x^2}=0$ Let $x=1$ and $y(1)=\frac{\pi}{4}$

$1+\left[1+\tan^2\frac{\pi}{4}\right]\cdot\frac{y'(1)+\frac{\pi}{4}}{1}=0 \Rightarrow 1+2.\left(y'(1)+\frac{\pi}{4}\right)=0$

$y'(1)=-\frac{1}{2}-\frac{\pi}{4}$ that is the slope of tangent line.

The equation of the tangent line is $y-\frac{\pi}{4}=\left(-\frac{1}{2}-\frac{\pi}{4}\right)(x-1)$

The equation of the normal line is $y-\frac{\pi}{4}=-\frac{1}{\left(-\frac{1}{2}-\frac{\pi}{4}\right)}(x-1)$

Inverse Functions

A function f is one-to-one if $f(x_1)\neq f(x_2)$ whenever $x_1\neq x_2$ or equivalently $f(x_1)=f(x_2)$ implies $x_1=x_2$.

If f is one-to-one, then it has an inverse function f^{-1}.

$y=f^{-1}(x) \Leftrightarrow x=f(y)$

Derivative of $y=f^{-1}(x) \Rightarrow x=f(y)$ by implicit differentiation

$$1=f'(y).\frac{dy}{dx}$$

$$\frac{dy}{dx}=\frac{1}{f'(y)} \Rightarrow \frac{d}{dx}f^{-1}(x)=\frac{1}{f'(f^{-1}(x))}$$

8. Show that the following functions are one-to-one and find $(f^{-1})'(1)$ and $(g^{-1})'(2\pi)$ where

(a) $f(x) = x^5 + x + 1$

(b) $g(x) = \sin x + 2x$

Solution:

(a) $f(x) = x^5 + x + 1 \Rightarrow f'(x) = x^4 + 1 > 0$ for all $x \in \mathbb{R}$ so f is strictly increasing; thus, it is one-to-one.

f has an inverse function.

$$(f^{-1})'(1) = \frac{1}{f'(f^{-1}(1))} = \frac{1}{f'(0)} = \frac{1}{1} = 1$$

$f(0) = 0 + 0 + 1 = 1 \Rightarrow f^{-1}(1) = 0$

(b) $g(x) = \sin x + 2x \Rightarrow g'(x) = \cos x + 2 > 0$ for all $x \in \mathbb{R}$ so g is strictly increasing; thus; it is one-to-one

$$(g^{-1})'(2\pi) = \frac{1}{g'(g^{-1}(2\pi))} = \frac{1}{g'(\pi)} = \frac{1}{\cos\pi + 2} = \frac{1}{-1+2} = 1$$

$g(\pi) = \sin\pi + 2\pi = 2\pi \Rightarrow g^{-1}(2\pi) = \pi$