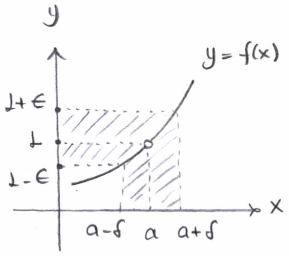
MATH 119 - CALCULUS WITH ANALYTIC GEOMETRY

RECITATION 3

A formal definition of limit :



$\lim_{x \to a} f(x) = L$ if the following condition is satisfied:

For every $\epsilon > 0$ there exists a $\delta > 0$ (depending on ϵ) such that if $0 < |x-a| < \delta$ then $|f(x) - L| < \epsilon$.

1. Use the formal $\epsilon - \delta$ definition of limit to prove $\lim_{x \to 3} (7x-6) = 15$.

Solution:

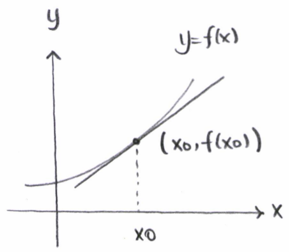
Let $\epsilon > 0$ be given. We need to find a $\delta > 0$ such that if $|x-3| < \delta$ then $|7x-6-15| < \epsilon$.

$|7x-21| < \epsilon \Rightarrow 7|x-3| < \epsilon \Rightarrow |x-3| < \frac{\epsilon}{7}$ Choose $\delta = \frac{\epsilon}{7}$

Thus; for every $\epsilon > 0$, choose $\delta = \frac{\epsilon}{7}$ if $|x-3| < \delta = \frac{\epsilon}{7}$ then $|f(x) - 15| < \epsilon$; that is,

$\lim_{x \to 3} (7x-6) = 15.$

Non-Vertical Tangent lines



Suppose f is continuous at $x = x_0$, $\lim_{h \to 0} \frac{f(x_0+h) - f(x_0)}{h} = m$ exists.

The straight line having slope m and passing through $(x_0, f(x_0))$ is called the tangent line to the graph of $y = f(x)$.

The equation: $y - f(x_0) = m(x - x_0)$.

2. Show that the tangent line to the graph of $f(x) = 12x - x^3$ at the point $(2,16)$ is horizontal.

Solution:

The slope of the tangent line;

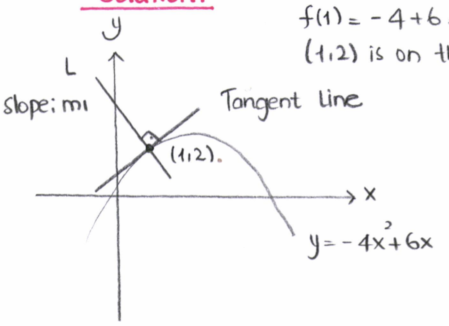
$$m = \lim_{h \to 0} \frac{f(2+h) - f(2)}{h} = \lim_{h \to 0} \frac{12(2+h) - (2+h)^3 - 16}{h} = \lim_{h \to 0} \frac{24 + 12h - h^3 - 6h^2 - 12h - 8 - 16}{h}$$

$$= \lim_{h \to 0} \frac{-h^3 - 6h^2}{h} = \lim_{h \to 0} (-h^2 - 6h) = 0$$

Since the slope is 0, this tangent line is horizontal.

3. Find an equation of the line that is perpendicular to the tangent line at the point $(1,2)$ to the graph of $f(x) = -4x^2 + 6x$.

Solution:



$f(1) = -4+6 = 2$

$(1,2)$ is on the curve !

The slope of the tangent line:

$$m = \lim_{h\to 0} \frac{f(1+h)-f(1)}{h} = \lim_{h\to 0} \frac{-4(1+h)^2+6(1+h)-2}{h}$$

$$= \lim_{h\to 0} \frac{-4-8h-4h^2+6+6h-2}{h}$$

$$= \lim_{h\to 0} \frac{-4h^2-2h}{h} = \lim_{h\to 0} (-4h-2) = -2$$

Since the line L and tangent line are perpendicular:

$m.m_1 = -1 \Rightarrow (-2).m_1 = -1 \Rightarrow m_1 = \frac{1}{2}$

The equation of the line : $y - 2 = \frac{1}{2}(x-1)$

Derivative: The derivative of a function f is defined by $f'(x) = \lim_{h\to 0} \frac{f(x+h)-f(x)}{h}$ at all points x for which the limit exists.

∇ The derivative at $x = x_0$: $f'(x_0) = \lim_{x\to x_0} \frac{f(x)-f(x_0)}{x-x_0}$ $\quad (h = x - x_0)$

4. Compute $f'(x)$ using the limit definition of derivative for the following functions:

(a) $f(x) = x^3$

(b) $f(x) = x - \sqrt{x}$.

Solution:

(a) $$f'(x) = \lim_{h\to 0} \frac{f(x+h)-f(x)}{h} = \lim_{h\to 0} \frac{(x+h)^3 - x^3}{h} = \lim_{h\to 0} \frac{\cancel{x^3} + 3x^2h + 3xh^2 + h^3 - \cancel{x^3}}{h}$$

$$= \lim_{h\to 0} (3x^2 + 3xh + h^2) = 3x^2$$

(b) $$f'(x) = \lim_{h\to 0} \frac{f(x+h)-f(x)}{h} = \lim_{h\to 0} \frac{(x+h) - \sqrt{x+h} - x + \sqrt{x}}{h} = \lim_{h\to 0} \frac{h + \sqrt{x} - \sqrt{x+h}}{h}$$

$$= \lim_{h\to 0} \left(1 + \frac{\sqrt{x}-\sqrt{x+h}}{h} \cdot \frac{\sqrt{x}+\sqrt{x+h}}{\sqrt{x}+\sqrt{x+h}}\right) = \lim_{h\to 0} \left(1 + \frac{\cancel{x} - \cancel{x} - \cancel{h}}{\cancel{h}.(\sqrt{x}+\sqrt{x+h})}\right)$$

$$= 1 - \frac{1}{2\sqrt{x}}$$

5. In the followings, each limit represents a derivative $f'(a)$. Find a function $f(x)$ and the point a.

(a) $\lim_{h\to 0} \dfrac{(5+h)^3 - 125}{h}$

(b) $\lim_{h\to 0} \dfrac{\sin\left(\frac{\pi}{6}+h\right) - \frac{1}{2}}{h}$

(c) $\lim_{x\to \frac{1}{4}} \dfrac{\frac{1}{x} - 4}{x - \frac{1}{4}}$

Solution:

(a) Let $f(x) = x^3$ and $a = 5$, f is differentiable at $x = 5$

$$\lim_{h\to 0} \frac{(5+h)^3 - 5^3}{h} = \lim_{h\to 0} \frac{f(5+h) - f(5)}{h} = f'(5).$$

(b) Let $f(x) = \sin x$, $a = \frac{\pi}{6}$ $\quad f\left(\frac{\pi}{6}\right) = \sin\frac{\pi}{6} = \frac{1}{2}$

$$\lim_{h\to 0} \frac{\sin\left(\frac{\pi}{6}+h\right) - \sin\frac{\pi}{6}}{h} = \lim_{h\to 0} \frac{f\left(h+\frac{\pi}{6}\right) - f\left(\frac{\pi}{6}\right)}{h} = f'\left(\frac{\pi}{6}\right)$$

(c) Let $f(x) = \frac{1}{x}$, $a = 4$ then $f\left(\frac{1}{4}\right) = 4$

$$\lim_{x\to \frac{1}{4}} \frac{\frac{1}{x} - 4}{x - \frac{1}{4}} = \lim_{x\to \frac{1}{4}} \frac{f(x) - f\left(\frac{1}{4}\right)}{x - \frac{1}{4}} = f'\left(\frac{1}{4}\right)$$

6. Let $f(x) = |x|$ and $g(x) = x.|x|$. Show that $f'(0)$ does not exists; however $g'(0)$ exists and equals to 0.

Solution:

$$f(x) = |x| = \begin{cases} x & \text{if } x \geq 0 \\ -x & \text{if } x < 0 \end{cases}$$

to find $f'(0)$, we need to find left and right derivatives at $x = 0$.

$$\left.\begin{aligned} f'_-(0) &= \lim_{h\to 0^-} \frac{f(0+h) - f(0)}{h} = \lim_{h\to 0^-} \frac{-h-0}{h} = -1 \\ f'_+(0) &= \lim_{h\to 0^+} \frac{f(0+h) - f(0)}{h} = \lim_{h\to 0^+} \frac{+h-0}{h} = 1 \end{aligned}\right\}$$

since $f'_-(0) \neq f'_+(0)$; $f'(0)$ does not exist.

$$g(x) = x|x| = \begin{cases} -x^2 & \text{if } x < 0 \\ x^2 & \text{if } x \geq 0 \end{cases}$$

Similarly; we will calculate left and right derivatives

$$\left.\begin{aligned} g'_-(0) &= \lim_{h\to 0^-} \frac{g(0+h) - g(0)}{h} = \lim_{h\to 0^-} \frac{-h^2-0}{h} = 0 \\ g'_+(0) &= \lim_{h\to 0^+} \frac{g(0+h) - g(0)}{h} = \lim_{h\to 0^+} \frac{h^2-0}{h} = 0 \end{aligned}\right\}$$

$g'_-(0) = g'_+(0) = 0$; so $g'(0) = 0$.

7. Find $f'(0)$ where $f(x) = \begin{cases} \dfrac{x^3 \cdot \sin\frac{1}{x}}{\sin x} & , \text{ if } x \neq 0 \\ 0 & , \text{ if } x = 0 \end{cases}$

Solution:

$$f'(0) = \lim_{h\to 0} \frac{f(0+h) - f(0)}{h} = \lim_{h\to 0} \frac{\dfrac{h^3 \cdot \sin\frac{1}{h}}{\sinh} - 0}{h} = \lim_{h\to 0} \frac{h^2 \cdot \sin\frac{1}{h}}{\sinh}$$

$$= \lim_{h\to 0} \left(h \cdot \sin\frac{1}{h}\right) \cdot \left(\frac{h}{\sinh}\right) = \lim_{h\to 0} \underbrace{\left(h \cdot \sin\frac{1}{h}\right)}_{(*)} \frac{1}{\underbrace{\frac{\sinh}{h}}_{\to 1 \text{ when } h\to 0}} = 0$$

(*) Since $-1 \leqslant \sin\frac{1}{h} \leqslant 1$; $\lim\limits_{h\to 0} (-h) = \lim\limits_{h\to 0} h = 0$ by Squeeze theorem $\lim\limits_{h\to 0} \left(h \cdot \sin\frac{1}{h}\right) = 0$

$-h \leqslant h \cdot \sin\frac{1}{h} \leqslant h$

Thus; $f'(0) = 0$

8. For the following functions; determine the points c (if any) such that $f'(c)$ does not exists.

(a) $f(x) = |x-2|$

(b) $f(x) = x^{3/2}$

(c) $f(x) = |x-2|^2$

Solution:

(a) $f(x) = |x-2| = \begin{cases} x-2 & \text{if } x \geqslant 2 \\ -x+2 & \text{if } x < 2 \end{cases}$

At the jump point check differentiability

$$f'_-(2) = \lim_{h\to 0^-} \frac{f(2+h) - f(2)}{h} = \lim_{h\to 0^-} \frac{-\cancel{2} - h + \cancel{2} - 0}{h} = -1$$

$$f'_+(2) = \lim_{h\to 0^+} \frac{f(2+h) - f(2)}{h} = \lim_{h\to 0^+} \frac{2+h-2-0}{h} = 1$$

$\left.\right\}$ $f'_-(2) \neq f'_+(2)$; $f'(2)$ does not exist.

For other real numbers; the function is differentiable.

(b) $f(x) = x^{3/2}$, the domain of this function $D(f) = [0, \infty)$

f is differentiable on its domain.

(c) $f(x) = |x-2|^2 = \begin{cases} (-x+2)^2 & \text{if} \quad x < 2 \\ (x-2)^2 & \text{if} \quad x \geqslant 2 \end{cases} = x^2 - 4x + 4$ so it is differentiable for all $x \in \mathbb{R}$.