

Math 120 Recitations

Week - 4

1. Determine the power series representations of the following functions. On what interval is each representation valid (converges to the value of the function)?

(a) $\frac{1}{120-x}$ in powers of x

(b) $\frac{1}{(120-x)^2}$ in powers of x

(c) $\ln x$ in powers of $x-120$

Soln: Note that $\frac{1}{1-x} = \sum_{n=0}^{\infty} x^n$ for $|x| < 1$

$$\begin{aligned} \text{Then } \frac{1}{120-x} &= \frac{1}{120(1-\frac{x}{120})} = \frac{1}{120} \cdot \frac{1}{1-\frac{x}{120}} \\ &= \frac{1}{120} \sum_{n=0}^{\infty} \left(\frac{x}{120}\right)^n \quad \text{for } \left|\frac{x}{120}\right| < 1 \\ &= \sum_{n=0}^{\infty} \frac{x^n}{120^{n+1}} \quad \text{for } |x| < 120 \end{aligned}$$

for the interval of convergence, check end points, re
 $x = \mp 120$

$$\text{if } x = 120 \Rightarrow \sum_{n=0}^{\infty} \frac{120^n}{120^{n+1}} = \sum_{n=0}^{\infty} \frac{120^n}{120^{n+1}} = \sum_{n=0}^{\infty} \frac{1}{120}$$

is divergent by n-th term test (since $\lim_{n \rightarrow \infty} \frac{1}{120} = \frac{1}{120} \neq 0$)

$$\text{if } x = -120 \Rightarrow \sum_{n=0}^{\infty} \frac{(-120)^n}{120^{n+1}} = \sum_{n=0}^{\infty} \frac{(-1)^n}{120} \quad \text{is divergent by n-th term test}$$

$$\left(\text{let } a_n = \frac{(-1)^n}{120} \Rightarrow \lim_{n \rightarrow \infty} a_{2n} = \frac{1}{120} \neq \lim_{n \rightarrow \infty} a_{2n+1} = -\frac{1}{120} \right.$$

$$\left. \Rightarrow \lim_{n \rightarrow \infty} \frac{(-1)^n}{120} \text{ dne.} \right)$$

$$\text{Hence, } \frac{1}{120-x} = \sum_{n=0}^{\infty} \frac{x^n}{120^{n+1}} \quad \text{for } |x| < 120$$

and interval of convergence: $(-120, 120)$

(b) We have $\frac{1}{120-x} = \sum_{n=0}^{\infty} \frac{x^n}{120^{n+1}}$ for $|x| < 120$
 from part (a).

Now, take the derivative of both sides, then
 we'll get

$$\frac{1}{(120-x)^2} = \sum_{n=1}^{\infty} \frac{n \cdot x^{n-1}}{120^{n+1}}, \quad |x| < 120$$

Consider the endpts:

$$\text{If } x=120 \Rightarrow \sum_{n=1}^{\infty} \frac{n \cdot x^{n-1}}{120^{n+1}} = \sum_{n=1}^{\infty} \frac{n \cdot 120^{n-1}}{120^{n+1}}$$

$$= \sum_{n=1}^{\infty} \frac{n}{120^2} \text{ is divergent}$$

by n-th term test.
 (since $\lim_{n \rightarrow \infty} \frac{n}{120^2} = \infty \neq 0$)

$$\text{If } x = -120 \Rightarrow \sum_{n=1}^{\infty} \frac{n \cdot x^{n-1}}{120^{n+1}} = \sum_{n=1}^{\infty} (-1)^{n-1} \cdot \frac{n}{120^2}$$

is divergent by n-th
term test.

(since $\lim_{n \rightarrow \infty} \frac{(-1)^n \cdot n}{120^2}$ does not exist)

$$\text{Hence, } \frac{1}{(120-x)^2} = \sum_{n=1}^{\infty} n \cdot \frac{x^{n-1}}{120^{n+1}} \text{ for } |x| < 120.$$

and interval of convergence: $(-120, 120)$

(c) Note that $\frac{1}{1-x} = \sum_{n=0}^{\infty} x^n$ for $|x| < 1$

$$\frac{1}{x} = \frac{1}{x-120+120} = \frac{1}{120 \left(1 + \frac{x-120}{120} \right)}$$

$$\stackrel{(*)}{=} \frac{1}{120} \sum_{n=0}^{\infty} \left(\frac{x-120}{120} \right)^n (-1)^n , \quad \left| \frac{x-120}{120} \right| < 1$$

↓
 $\left(\equiv 0 < x < 240 \right)$

Take the integral of both sides: (of $(*)$):

$$\Rightarrow \ln|x| = C + \sum_{n=0}^{\infty} \frac{1}{120^{n+1}} \cdot \frac{(x-120)^{n+1}}{n+1} (-1)^n$$

↑
 constant.

find the constant C :

$$\text{Since } \ln|x| = C + \sum_{n=0}^{\infty} \frac{(-1)^n (x-120)^{n+1}}{120^{n+1}} \quad \text{for } 0 < x < 240$$

$$\text{take } x=120 \stackrel{c \in (0, 240)}{\Rightarrow} \ln(120) = C + \sum_{n=0}^{\infty} 0 \Rightarrow C = \ln(120)$$

↑
 center

Now, check endpoints for the interval of convergence

if $x=0$:

$$\sum_{n=0}^{\infty} \frac{1}{120^{n+1}} \frac{1}{n+1} \cdot (x-120)^{n+1} (-1)^n = \sum_{n=0}^{\infty} \frac{(-120)^{n+1}}{120^{n+1} (n+1)} (-1)^n$$

$$\begin{aligned}
 &= \sum_{n=0}^{\infty} \frac{-1}{n+1} \\
 &= - \left(\sum_{n=0}^{\infty} \frac{1}{n+1} \right) \\
 &\quad \text{by divergent by LCT (with } \sum_{n=1}^{\infty} \frac{1}{n} \text{)} \\
 &- \sum_{n=0}^{\infty} \frac{1}{n+1} \text{ is also divergent}
 \end{aligned}$$

If $x = 240$:

$$\sum_{n=0}^{\infty} \frac{1}{120^{n+1}} \cdot \frac{1}{n+1} \cdot (x-120)^{n+1} (-1)^n = \sum_{n=0}^{\infty} \frac{(-1)^n}{n+1} \text{ is conv by AST.}$$

Te, $\ln(120) + \sum_{n=0}^{\infty} \frac{(-1)^n}{n+1}$ is convergent

$$\text{Hence, } \ln(x) = \ln(120) + \sum_{n=0}^{\infty} \frac{1}{120^{n+1}} \cdot \frac{1}{n+1} \cdot (x-120)^{n+1} (-1)^n$$

for $x \in (0, 240]$

↳ interval of convergence.

2. For each of the following series, find the sum.

(a) $\sum_{n=3}^{\infty} \frac{1}{n2^n}$.

(b) $\sum_{n=2}^{\infty} \frac{n^2}{2^n}$.

(a) We have $\frac{1}{1-x} = \sum_{n=0}^{\infty} x^n$ for $|x| < 1$.

Let take the integral of both sides,

$$\int \frac{1}{1-x} dx = \int \left(\sum_{n=0}^{\infty} x^n \right) dx$$

$$\Rightarrow -\ln|1-x| = C + \sum_{n=0}^{\infty} \frac{x^{n+1}}{n+1} \quad \text{for } |x| < 1$$

for some $C \in \mathbb{R}$.

$$\text{if } x=0 \in (-1, 1) \Rightarrow -\ln|1-0| = C + \sum_{n=0}^{\infty} 0 = C$$

\downarrow
 $C=0$

Hence,

$$-\ln(1-x) = \sum_{n=1}^{\infty} \frac{x^n}{n} \quad \text{for } |x| < 1$$

$$\text{if } x = \frac{1}{2} \in (-1, 1) \text{ then } -\ln\left(1 - \frac{1}{2}\right) = \sum_{n=1}^{\infty} \frac{1}{n \cdot 2^n}$$

$$\Rightarrow -\ln\left(\frac{1}{2}\right) = \frac{1}{2} + \frac{1}{2 \cdot 2^2} + \sum_{n=3}^{\infty} \frac{1}{n \cdot 2^n}$$

$$\Rightarrow \sum_{n=3}^{\infty} \frac{1}{n \cdot 2^n} = \ln(2) - \frac{5}{8}$$

(b) We know $\frac{1}{1-x} = \sum_{n=0}^{\infty} x^n$ for $|x| < 1$

Take the derivative of both sides (term by term)

$$\left(\frac{1}{1-x}\right)^2 = \sum_{n=1}^{\infty} n \cdot x^{n-1} \quad \text{for } |x| < 1$$

$$\Rightarrow x \cdot \frac{1}{(1-x)^2} = x \cdot \sum_{n=1}^{\infty} n \cdot x^{n-1} = \sum_{n=1}^{\infty} n \cdot x^n \quad \text{for } |x| < 1$$

Take the derivative of both sides,

$$\frac{(1-x)x' + x \cdot 2(1-x)'}{(1-x)^3} = \sum_{n=1}^{\infty} n^2 \cdot x^{n-1} \quad \text{for } |x| < 1$$

Multiply by x ,

$$x \cdot \frac{1-x+2x}{(1-x)^3} = \sum_{n=1}^{\infty} n^2 \cdot x^n \quad \text{for } |x| < 1$$

$$\text{if } x = \frac{1}{2} < 1 \Rightarrow \frac{1}{2} \cdot \frac{1+\frac{1}{2}}{\left(1-\frac{1}{2}\right)^3} = \sum_{n=1}^{\infty} \frac{n^2}{2^n}$$

$$\Rightarrow \frac{1}{2} \cdot \frac{3}{2} \cdot 8 = \frac{1}{2} + \sum_{n=2}^{\infty} \frac{n^2}{2^n}$$

$$\Rightarrow \sum_{n=2}^{\infty} \frac{n^2}{2^n} = 6 - \frac{1}{2} = \frac{11}{2}$$

3. Determine the first three terms of the Maclaurin series of $f(x) = (1+x)^\alpha$, where $\alpha \in \mathbb{R}$.

Soln:

Note: If f has derivatives of all orders at $x=c$ then

the series $\sum_{k=0}^{\infty} \frac{f^{(k)}(c)}{k!} (x-c)^k = f(c) + f'(c)(x-c) + \dots$

is called Taylor series of f about c .

If $c=0$ then the series called Maclaurin series.

We have $f(x) = (x+1)^\alpha$, $\alpha \in \mathbb{R}$

Note that Maclaurin series of f :

$$\sum_{n=0}^{\infty} \frac{f^{(n)}(0)}{n!} (x)^n = f(0) + \underbrace{f'(0)x + \frac{f''(0)}{2!}x^2 + \frac{f'''(0)}{3!}x^3}_{\text{first 3 terms (*)}} + \dots$$

where

- $f(0) = 1$
- $f'(x) = \alpha (x+1)^{\alpha-1} \Rightarrow f'(0) = \alpha$
- $f''(x) = \alpha(\alpha-1) (x+1)^{\alpha-2} \Rightarrow f''(0) = \alpha(\alpha-1)$
- $f'''(x) = \alpha(\alpha-1)(\alpha-2) (x+1)^{\alpha-3} \Rightarrow f'''(0) = \alpha(\alpha-1)(\alpha-2)$

⋮

$$\bullet f^{(k)}(x) = \alpha(\alpha-1)(\alpha-2)\dots(\alpha-(k-1)) (x+1)^{\alpha-k}$$

$$\text{and } f^{(k)}(0) = \alpha(\alpha-1)(\alpha-2)\dots(\alpha-k+1)$$

Then the Maclaurin series of f is

$$\sum_{n=0}^{\infty} \frac{\alpha(\alpha-1)(\alpha-2)\dots(\alpha-n+1)}{n!} x^n$$

$$= 1 + \underbrace{\frac{\alpha}{1!}x + \frac{\alpha(\alpha-1)}{2!}x^2}_{\text{first 3 terms}} + \frac{\alpha(\alpha-1)(\alpha-2)}{3!}x^3 + \dots$$

for $\alpha \in \mathbb{R}$.

Note: If α is a positive integer then the Maclaurin series of f will be the Binomial expansion.

2. Starting from the formula $\frac{1}{1-x} = \sum_{n=0}^{\infty} x^n$ for $|x| < 1$, find the Maclaurin series of the following functions, and determine their intervals of convergence:

$$(a) f(x) = \frac{3}{x^2 + 4}$$

$$(b) f(x) = x \ln(1 + 2x)$$

$$(c) f(x) = \arctan(x) \quad \text{Answer: } f(x) = \sum_{n=0}^{\infty} \frac{(-1)^n}{2n+1} x^{2n+1} \text{ for } |x| \leq 1$$

Soln: (a)

$$\frac{1}{1+x^2} = \sum_{n=0}^{\infty} (-1)^n x^{2n} \text{ for } |x^2| < 1$$

$$f(x) = \frac{3}{4+x^2} = \frac{3}{4(1+\frac{x^2}{4})} = \frac{3}{4} \cdot \frac{1}{1+(\frac{x}{2})^2}$$

$$= \frac{3}{4} \sum_{n=0}^{\infty} (-1)^n \left(\frac{x}{2}\right)^{2n} \text{ for } \left|\frac{x^2}{4}\right| < 1 \quad \begin{matrix} \downarrow \\ |x^2| < 4 \\ |x| < 2 \end{matrix}$$

$$\Rightarrow f(x) = \sum_{n=0}^{\infty} 3 (-1)^n \frac{x^{2n}}{2^{2n+2}} \text{ for } |x| < 2$$

Check end points: $x = \pm 2$

If $x=2 \Rightarrow \frac{3}{4} \sum (-1)^n \left(\frac{x}{2}\right)^{2n} = \frac{3}{4} \sum_{n=0}^{\infty} (-1)^n$ is divergent by n-th term test

If $x=-2 \Rightarrow \frac{3}{4} \sum (-1)^n \left(\frac{x}{2}\right)^{2n} = \frac{3}{4} \sum_{n=0}^{\infty} (-1)^n$ is divergent

Hence, $f(x) = \frac{3}{4+x^2} = \left(\sum_{n=0}^{\infty} (-1)^n \cdot 3 \cdot \frac{x^{2n}}{2^{2n+2}} \right)$ for $|x| < 2$

i.e., interval of convergence: $(-2, 2)$. Maclaurin series of f .

$$(b) f(x) = x \ln(1+2x)$$

Note that $\frac{1}{1+x} = \sum_{n=0}^{\infty} (-1)^n x^n, |x| < 1$

$$\Rightarrow \frac{1}{1+2x} = \sum_{n=0}^{\infty} (-1)^n (2x)^n, |2x| < 1$$

Take the integral of both sides:

$$\ln|1+2x| \cdot \frac{1}{2} = C + \sum_{n=0}^{\infty} (-1)^n \cdot 2^n \cdot \frac{x^{n+1}}{n+1}, |2x| < 1.$$

for some $C \in \mathbb{R}$.

$$\text{if } x=0 \in (-\frac{1}{2}, \frac{1}{2}) \Rightarrow \ln|1+0| \cdot \frac{1}{2} = C + \sum_{n=0}^{\infty} 0 = C$$

$\boxed{C=0}$

Check endpoints: $x = \pm \frac{1}{2}$

$$\text{if } x = \frac{1}{2} \Rightarrow \sum_{n=0}^{\infty} (-1)^n \cdot 2^{n+1} \cdot \frac{\left(\frac{1}{2}\right)^{n+2}}{n+1} = \sum_{n=0}^{\infty} \frac{(-1)^n \cdot 2^{n+1}}{2^{n+2} \cdot (n+1)}$$

$$= \sum_{n=0}^{\infty} \frac{(-1)^n}{2^{n+2}} \text{ is convergent by AST.}$$

$$\text{if } x = -\frac{1}{2} \Rightarrow \sum_{n=0}^{\infty} (-1)^n \cdot 2^{n+1} \cdot \frac{\left(-\frac{1}{2}\right)^{n+2}}{n+1} = \sum_{n=0}^{\infty} \frac{(-1)^n \cdot 2^{n+1} \cdot \frac{(-1)^{n+2}}{2^{n+2} \cdot (n+1)}}{n+1}$$

$$= \sum_{n=0}^{\infty} \frac{1}{2^{n+2}} \text{ is divergent by comparison test (with } \sum_{n=0}^{\infty} \frac{1}{n})$$

Hence, $f(x) = x \ln(1+2x) = \sum_{n=0}^{\infty} (-1)^n \cdot 2^{n+1} \frac{x^{n+2}}{n+1}$ for $x \in (-\frac{1}{2}, \frac{1}{2}]$

ie,

Interval of convergence: $(-\frac{1}{2}, \frac{1}{2}]$ and $\sum_{n=0}^{\infty} (-1)^n \cdot 2^{n+1} \frac{x^{n+2}}{n+1}$
is the Maclaurin series of $f(x) = x \ln(1+2x)$.

(c) $f(x) = \arctan(x)$

We have $\frac{1}{1-x} = \sum_{n=0}^{\infty} x^n$ - $|x| < 1$

Then, $\frac{1}{1+x^2} = \sum_{n=0}^{\infty} (-1)^n x^{2n}$ for $|x^2| < 1$.

Take the integral of both sides:

$$\arctan(x) = c + \sum_{n=0}^{\infty} (-1)^n \frac{x^{2n+1}}{2n+1}, \quad |x| < 1$$

if $x=0 \in (-1, 1)$, $\Rightarrow \arctan(0) = c + \sum_{n=0}^{\infty} 0 \Rightarrow c=0$

Check endpoints: $x=\pm 1$

if $x=1 \Rightarrow \sum_{n=0}^{\infty} (-1)^n \frac{x^{2n+1}}{2n+1} = \sum_{n=0}^{\infty} (-1)^n \frac{1}{2n+1}$ is convergent
by AST.

if $x=-1 \Rightarrow \sum_{n=0}^{\infty} (-1)^n \frac{x^{2n+1}}{2n+1} = \sum_{n=0}^{\infty} (-1)^n (-1)^{2n+1} \cdot \frac{1}{2n+1}$
 $= \sum_{n=0}^{\infty} \frac{(-1)^{n+1}}{2n+1}$ is conv by AST.

$$\text{Hence, } f(x) = \arctan(x) = \sum_{n=0}^{\infty} (-1)^n \frac{x^{2n+1}}{2n+1} \quad \text{for } |x| \leq 1.$$

ie,

$$\text{interval of convergence: } [-1, 1] \text{ and } \sum_{n=0}^{\infty} (-1)^n \frac{x^{2n+1}}{2n+1}$$

is the Maclaurin series of $f(x) = \arctan(x)$