

Math 120 Recitations

Week - 03

1. Determine whether the following series converge absolutely, converge conditionally, or diverge.

(a) $\sum_{n=1}^{\infty} \frac{(-1)^n}{n^{3/2}}.$

(b) $\sum_{n=1}^{\infty} \frac{(-1)^n}{n^{1/2}}.$

(c) $\sum_{n=1}^{\infty} \frac{\cos(n\pi)(n^3 - 3n^2 + 7)}{2n^3 + 13}.$

Soln: (a)

Note: $\sum_{n=1}^{\infty} |a_n|$ is absolutely convergent if $\sum_{n=1}^{\infty} |a_n|$ is convergent.

Let $a_n = \frac{(-1)^n}{n^{3/2}}$, then consider $\sum_{n=1}^{\infty} |a_n|$

where $|a_n| = \frac{1}{n^{3/2}}$

$\sum_{n=1}^{\infty} \frac{1}{n^{3/2}}$ is convergent by p-series test ($p = 3/2 > 1$)

then $\sum_{n=1}^{\infty} \frac{(-1)^n}{n^{3/2}}$ is absolutely convergent

(b) Let $a_n = \frac{(-1)^n}{n^{1/2}}$ then $\sum_{n=1}^{\infty} |a_n| = \sum_{n=1}^{\infty} \frac{1}{n^{1/2}}$ is divergent by p-series test. ($p = 1/2 < 1$)

$\Rightarrow \sum_{n=1}^{\infty} a_n$ is not absolutely convergent.

Note: The alternating series test:

Suppose $\{a_n\}$ is a sequence whose terms satisfy, for some positive integer N ,

$$(1) \quad a_n \text{ and } (-1)^n < 0 \quad \forall n \geq N$$

$$(2) \quad |(-1)^n a_{n+1}| \leq |a_n| \text{ for } n \geq N$$

$$(3) \quad \lim_{n \rightarrow \infty} a_n = 0$$

Then the series $\sum_{n=1}^{\infty} a_n$ converges.

We have $a_n = \frac{(-1)^n}{n^{1/2}}$

$$(1) \quad (-1)^{n+1} \cdot a_n < 0 \quad \forall n \geq 1$$

$$(2) \quad \text{Since } |a_n| = \frac{1}{\sqrt{n}} \quad \text{and} \quad \sqrt{n} \leq \sqrt{n+1} \quad \forall n \geq 1$$

$$\text{we have } |(-1)^{n+1}| = \frac{1}{\sqrt{n+1}} \leq |a_n| = \frac{1}{\sqrt{n}} \quad \forall n \geq 1$$

$$(3) \quad \text{Since } \lim_{n \rightarrow \infty} |a_n| = \lim_{n \rightarrow \infty} \frac{1}{\sqrt{n}} = 0, \quad \text{we have}$$

$$\lim_{n \rightarrow \infty} a_n = 0.$$

So by Alternating series test, $\sum_{n=1}^{\infty} a_n$ is convergent

$\Rightarrow \sum_{n=1}^{\infty} \frac{(-1)^n}{n^{1/2}}$ is conditionally convergent.

Note: $\sum_{n=1}^{\infty} a_n$ conditionally conv if $\sum_{n=1}^{\infty} |a_n|$ is divergent

but $\sum_{n=1}^{\infty} |a_n|$ is divergent.

$$(C) \text{ Let } a_n = \frac{\cos(n\pi)(n^3 - 3n^2 + 7)}{2n^3 + 13} = \frac{(-1)^n(n^3 - 3n^2 + 7)}{2n^3 + 13}$$

$$\text{Consider } a_{2n} = \frac{(2n)^3 - 3(2n)^2 + 7}{2(2n)^3 + 13}$$

$$\text{and } \lim_{n \rightarrow \infty} a_{2n} = \frac{1}{2}$$

$$\text{Also consider } a_{2n+1} \text{ and } \lim_{n \rightarrow \infty} a_{2n+1} = -\frac{1}{2}$$

$\Rightarrow \lim_{n \rightarrow \infty} a_n$ dne, hence $\sum_{n=1}^{\infty} a_n$ is divergent by n-th term test.

Note: (3). condition of AST is not satisfied for the series $\sum_{n=1}^{\infty} a_n$. So AST says nothing about convergence / divergence of $\sum_{n=1}^{\infty} a_n$.

2. For the following series, find the smallest integer n that ensures that the partial sum s_n approximates the sum s of the series with $|error| = |s - s_n| < 0.001$.

$$(a) \sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{(2n)!}$$

$$(b) \sum_{n=1}^{\infty} \frac{(-1)^n}{n \cdot \log_2(n)}$$

Soln: (a)

Note: Error Estimate for Alternating Series:

If the sequence $\{a_n\}$ satisfies the conditions of AST, so that the series $\sum_{n=1}^{\infty} a_n$ converges to the sum s . Then the error in the approximation $s \approx s_n$ (where $n > N$) has the same sign as the first omitted term $a_{n+1} = s_{n+1} - s_n$ and its size is no greater than the size of that term.

$$|s - s_n| \leq |s_{n+1} - s_n| = |a_{n+1}|$$

$$\text{We have } a_n = \frac{(-1)^{n-1}}{(2n)!}$$

Consider the AST:

$$(1) \quad a_{n+1} \cdot a_n < 0 \quad \forall n \geq 1$$

$$(2) \quad |a_{n+1}| = \frac{1}{(2n)!}, \quad \text{since } (2(n+1))! > (2n)! \quad \forall n \geq 1$$

$$\text{we have } |a_{n+1}| = \frac{1}{(2n+2)!} \leq |a_n| = \frac{1}{(2n)!} \quad \forall n \geq 1$$

$$(3) \quad \text{Since } \lim_{n \rightarrow \infty} |a_n| = \lim_{n \rightarrow \infty} \frac{1}{(2n)!} = 0, \quad \lim_{n \rightarrow \infty} a_n = 0.$$

Hence $\sum_{n=1}^{\infty} a_n$ is convergent by AST.

Suppose $\sum_{n=1}^{\infty} a_n$ converges to s .

By alternating series error estimation,

$$|\text{error}| = |s - s_n| \leq |s_{n+1} - s_n| = |a_{n+1}| = \frac{1}{(2n+2)!}$$

$$\frac{1}{(2n+2)!} \leq 10^{-3} = \frac{1}{1000}$$

$$\Rightarrow 1000 \leq (2n+2)! \quad \text{Note: } \frac{b!}{7!} = 720$$

$$\text{if } 2n+2 \geq 7 \text{ then } 1000 \leq (2n+2)!$$

So the smallest n=3

ie,

If $n=3$ then $|\text{error}| \leq 0.001$

$$\Rightarrow \sum_{n=1}^{\infty} a_n \approx a_1 + a_2 + a_3 = \frac{1}{2!} - \frac{1}{4!} + \frac{1}{6!}$$

(b) Let $a_n = \frac{(-1)^n}{n \log_2(n)}$ $\forall n \geq 1$.

Consider the AST:

$$(1) a_{n+1} - a_n < 0 \quad \forall n \geq 1$$

$$(2) |a_n| = \frac{1}{n \log_2^n} \quad \forall n \geq 1 \quad \text{and} \quad (n+1) \log_2^{(n+1)} > n \log_2^n \quad \forall n \geq 1.$$

we have $|a_{n+1}| \leq |a_n| \quad \forall n \geq 1.$

(3) Since $\lim_{n \rightarrow \infty} |a_n| = \lim_{n \rightarrow \infty} \frac{1}{n \log_2^n} = 0$, we have

$$\lim_{n \rightarrow \infty} a_n = 0.$$

so by AST, $\sum_{n=1}^{\infty} a_n$ converges.

$$|\text{error}| = |s - s_n| \leq |a_{n+1}| = \frac{1}{(n+1) \log_2^{(n+1)}} \leq \frac{1}{1000}$$

$$\Rightarrow 1000 \leq (n+1) \log_2^{(n+1)} \quad \underline{\text{Note:}}$$

$$141 \cdot \log_2 141 \approx 1007.$$

so if $n+1 \geq 141$ then $1000 \leq (n+1) \log_2^{(n+1)}$.

i.e., the smallest $n = 140$

re,
if $n = 140$ then $\sum_{n=1}^{\infty} a_n \approx \sum_{n=1}^{140} a_n$ with $|\text{error}| \leq 10^{-3}$

3. For which values of x does the series $\sum_{n=1}^{\infty} \frac{(2\sin(x))^n}{n}$ converge?

Soln: Let $a_n = \frac{(2\sin(x))^n}{n}$ for some $x \in \mathbb{R}$.

$$\begin{aligned} \text{consider } \lim_{n \rightarrow \infty} \left| \frac{a_{n+1}}{a_n} \right| &= \lim_{n \rightarrow \infty} \left| \frac{(2\sin(x))^{n+1}}{(n+1)} \cdot \frac{n}{(2\sin(x))^n} \right| \\ &= \lim_{n \rightarrow \infty} \left| 2\sin x \cdot \frac{n}{n+1} \right| = |2\sin(x)| \end{aligned}$$

so if $|2\sin(x)| < 1$ then the series converges absolutely

i.e., if $|\sin(x)| < \frac{1}{2}$ " " "

$$-\frac{1}{2} < \sin(x) < \frac{1}{2}$$

$$\Rightarrow x \in \left(-\frac{\pi}{6} + 2k\pi, \frac{\pi}{6} + 2k\pi\right) \cup \left(-\frac{\pi}{6} + (2k+1)\pi, \frac{\pi}{6} + (2k+1)\pi\right)$$

then $\sum_{n=1}^{\infty} \left| \frac{(2\sin(x))^n}{n} \right|$ is (abs) convergent.

Now let consider the $(-\frac{\pi}{6}, \frac{\pi}{6}) \cup (\frac{5\pi}{6}, \frac{7\pi}{6})$

and check endpoints for this set

Check end pts:

if $x = -\frac{\pi}{6} \Rightarrow \sum_{n=1}^{\infty} \left| \frac{(2\sin(x))^n}{n} \right| = \sum_{n=1}^{\infty} \frac{(-1)^n}{n}$ is conv by AST.

If $x = \frac{\pi}{6}$ $\Rightarrow \sum_{n=1}^{\infty} \left(\frac{2\sin(x)}{n}\right)^n = \sum_{n=1}^{\infty} \frac{1}{n}$ is divergent
by p-series test

If $x = \frac{5\pi}{6}$ $\Rightarrow \sum_{n=1}^{\infty} \left(\frac{2\sin(x)}{n}\right)^n = \sum_{n=1}^{\infty} \frac{1}{n}$ is div.

If $x = \frac{7\pi}{6}$ $\Rightarrow \sum_{n=1}^{\infty} \left(\frac{2\sin(x)}{n}\right)^n = \sum_{n=1}^{\infty} \frac{(-1)^n}{n}$ is conv.

So $x \in \left[-\frac{\pi}{6}, \frac{\pi}{6}\right] \cup \left[\frac{5\pi}{6}, \frac{7\pi}{6}\right]$ then $\sum_{n=1}^{\infty} a_n$ is conv.

Since $\sin(x)$ is 2π -periodic, we have

$$x \in \left[-\frac{\pi}{6} + 2\pi k, \frac{\pi}{6} + 2\pi k\right) \cup \left(-\frac{\pi}{6} + (2k+1)\pi, \frac{\pi}{6} + (2k+1)\pi\right]$$

for $k \in \mathbb{Z}$ then $\sum_{n=1}^{\infty} a_n$ is convergent.

4. Determine the center, radius and interval of convergence of each of the following series.

(a) $\sum_{n=1}^{\infty} \frac{(2x+3)^n}{n^{120}}$.

(b) $\sum_{n=1}^{\infty} \frac{e^n}{n!} \cdot (x-120)^n$.

(c) $\sum_{n=1}^{\infty} n^n \cdot (x-2022)^n$.

Soln: (a) Let $a_n = \frac{(2x+3)^n}{n^{120}}$ for some $x \in \mathbb{R}$.

Note that if $x = -\frac{3}{2}$ then $\sum_{n=1}^{\infty} a_n = \sum_{n=1}^{\infty} 0 = \lim_{n \rightarrow \infty} s_n = \lim_{n \rightarrow \infty} 0 = 0$

ie, if $x = -\frac{3}{2}$ then $\sum_{n=1}^{\infty} \frac{(2x+3)^n}{n^{120}}$ is convergent.

Now consider the series $\sum_{n=1}^{\infty} |a_n| = \sum_{n=1}^{\infty} \left| \frac{(2x+3)^n}{n^{120}} \right|$

Note that $\{|a_n|\}_{n=1}^{\infty}$ is a positive sequence $\forall x \neq -\frac{3}{2}$

so we can apply the ratio test.

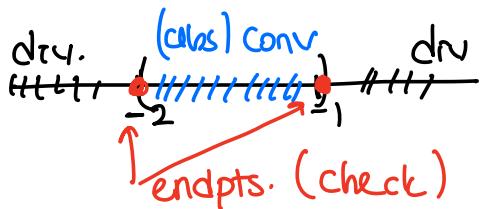
$$\text{Consider } \lim_{n \rightarrow \infty} \left| \frac{a_{n+1}}{a_n} \right| = \lim_{n \rightarrow \infty} \left| \frac{(2x+3)^{n+1}}{(n+1)^{120}} \cdot \frac{n^{120}}{(2x+3)^n} \right|, x \neq -\frac{3}{2}$$

$$= \lim_{n \rightarrow \infty} |2x+3| \left(\frac{n}{n+1} \right)^{120} = |2x+3|$$

If $|2x+3| < 1$ then $\sum_{n=1}^{\infty} a_n$ converges absolutely.
(abs conv \Rightarrow conv)

ie, $-2 < x < -1$ " " " "

So if $x \in (-2, -1)$ then $\sum_{n=1}^{\infty} \frac{(2x+3)^n}{n^{120}}$ is (abs) convergent.



for the interval of convergence, consider the endpts, i.e., $x = -1$ and $x = -2$

If $x = -1 \Rightarrow \sum_{n=1}^{\infty} c_n = \sum_{n=1}^{\infty} \frac{1}{n^{120}}$ is convergent by p-series test ($p = 120 > 1$)

if $x = -2 \Rightarrow \sum_{n=1}^{\infty} c_n = \sum_{n=1}^{\infty} \frac{(-1)^n}{n^{120}}$ is convergent by AST.

Hence, interval of convergence $I = [-2, -1]$

\Rightarrow radius of convergence := $r = \frac{1}{2}$

and center: $x = -1, \text{ i.e. } = -\frac{3}{2}$

(b) Let $c_n = \frac{e^n}{n!} (x-120)^n$

Note that if $x = 120$ then $\sum_{n=1}^{\infty} \frac{e^n}{n!} (x-120)^n$ is convergent.

Consider the series $\sum_{n=1}^{\infty} \left(\frac{e^n}{n!} (x-120)^n \right)$

$$\text{Then, } \lim_{n \rightarrow \infty} \left| \frac{a_{n+1}}{a_n} \right| = \lim_{n \rightarrow \infty} \frac{e^{n+1}}{(n+1)!} \cdot \frac{n!}{e^n} |x - 120|, \quad x \neq 120$$

$$= \lim_{n \rightarrow \infty} \frac{e}{n+1} |x - 120|$$

$$= 0 \in [0, 1)$$

So by the ratio test $\sum_{n=1}^{\infty} |a_n|$ is convergent $\forall x \in \mathbb{R}$

Hence, $\sum_{n=1}^{\infty} a_n$ is convergent $\forall x \in \mathbb{R}$.
 (abs conv \Rightarrow conv)

Interval of convergence is $(-\infty, \infty)$ and the radius of convergence is ∞ .

Center: $x = 120$.

(c) let $a_n = n^n (x - 2022)^n$

Note that if $x = 2022$ then

$$\sum_{n=1}^{\infty} n^n (x - 2022)^n = \sum_{n=1}^{\infty} 0 = \lim_{n \rightarrow \infty} s_n = \lim_{n \rightarrow \infty} 0 = 0$$

Now consider $\sum_{n=1}^{\infty} |n^n (x - 2022)^n|$,

$$\text{then } \lim_{n \rightarrow \infty} \left\| \left(n^n (x - 2022)^n \right)^{\frac{1}{n}} \right\| \stackrel{x \neq 2022}{=} \lim_{n \rightarrow \infty} |n \cdot (x - 2022)| = \infty$$

Hence by root test $\sum_{n=1}^{\infty} |n^n (x-2022)^n|, x \neq 2022$

is divergent

Hence, $\sum_{n=1}^{\infty} n^n (x-2022)^n$ diverges $\forall x \in \mathbb{R} - \{2022\}$

i.e., $I = \{2022\}$ interval of conv, radius = $r = 0$.

Center: $x = 2022$