

Math 120 Recitation

Week-2

(1) Let $\sum_{n=1}^{\infty} a_n$ be a convergent series. What can you say about convergence of the given series by using n-th term test.

Note: nth term test

If $\sum a_n$ conv then $\lim_{n \rightarrow \infty} a_n = 0$.

Note that n-th term test implies the following:

If $\lim_{n \rightarrow \infty} a_n \neq 0$ then $\sum_{n=1}^{\infty} a_n$ is divergent.

[If $\lim_{n \rightarrow \infty} a_n = 0 \Rightarrow \sum_{n=1}^{\infty} a_n$ conv, for exp: $\sum_{n=1}^{\infty} \frac{1}{n}$]

$$(a) \sum_{n=1}^{\infty} \frac{1+a_n}{3-a_n}, \quad a_n \neq 3 \quad \forall n \geq 1.$$

Let consider the series $\sum_{n=1}^{\infty} b_n$ where $b_n = \frac{1+a_n}{3-a_n} \quad \forall n \geq 1$

Note that $\lim_{n \rightarrow \infty} a_n = 0$ since $\sum_{n=1}^{\infty} a_n$ converges.

$$\text{Consider } \lim_{n \rightarrow \infty} b_n = \lim_{n \rightarrow \infty} \frac{1+a_n}{3-a_n} = \frac{\lim_{n \rightarrow \infty} 1+a_n}{\lim_{n \rightarrow \infty} 3-a_n} \text{, } \lim_{n \rightarrow \infty} a_n = 0.$$

$$= \frac{1}{3} \neq 0$$

i.e., $\sum_{n=1}^{\infty} b_n$ is divergent. by nth term test.

$$(b) \sum_{n=1}^{\infty} \sin\left(\frac{\pi a_n}{2}\right)$$

We have $\sum_{n=1}^{\infty} a_n$ converges, i.e., $\lim_{n \rightarrow \infty} a_n = 0$.

Let $b_n = \sin\left(\frac{\pi a_n}{2}\right) \quad \forall n \geq 1$. Then we have the series $\sum_{n=1}^{\infty} b_n$

$$\lim_{n \rightarrow \infty} b_n = \lim_{n \rightarrow \infty} \sin\left(\frac{\pi a_n}{2}\right) = \sin\left(\lim_{n \rightarrow \infty} \frac{\pi a_n}{2}\right) = 0$$

↑
"sin cont"

\Rightarrow The nth term test gives no information.

Let consider the case when $a_n > 0 \quad \forall n \geq 1$.

If $a_n > 0 \quad \forall n \geq 1$, then we have

$$b_n := \sin\left(\frac{\pi a_n}{2}\right) \leq \frac{\pi a_n}{2} := c_n$$

(Note: $\sin x \leq x \quad \forall x \geq 0$)

Note that $\sin\left(\frac{\pi a_n}{2}\right) > 0$ when $n \rightarrow \infty$ ($a_n > 0 \quad \forall n \geq 1$ since $\lim_{n \rightarrow \infty} a_n = 0$)

Since $\sum_{n=1}^{\infty} \frac{\pi a_n}{2}$ is convergent, by CT

$\sum_{n=1}^{\infty} \sin\left(\frac{\pi a_n}{2}\right)$ is also conv. if $a_n > 0 \quad \forall n \geq 1$.

Note: Comparison Test (CT)

if $0 \leq a_n \leq b_n \cdot L$, $L \in \mathbb{R}^+$

- if $\sum b_n$ conv $\Rightarrow \sum a_n$ conv
- if $\sum a_n$ div $\Rightarrow \sum b_n$ div

2 Find the sum of the following series, if it converges.

$$(a) \sum_{n=3}^{\infty} \frac{3^{n-1}}{5 \cdot 4^{n+3}},$$

$$(d) \sum_{n=1}^{\infty} \frac{1}{n^7 + n}$$

$$(b) \sum_{n=1}^{\infty} \sin^{2n} \theta, \text{ for } 0 < \theta < \pi/2,$$

$$(e) \sum_{n=1}^{\infty} \frac{1}{n^2 + 2n}$$

$$(c) \sum_{n=0}^{\infty} \frac{2^{2n+1}}{\pi^{n-2}}.$$

Soln: (a) $\sum_{n=3}^{\infty} a_n$ where $a_n = \frac{3^{n-1}}{5 \cdot 4^{n+3}}$

Note: Geometric series

A series of the form $\sum_{n=1}^{\infty} ar^{n-1} = a + ar + ar^2 + \dots$

whose n th term is $a_n = ar^{n-1}$ is called geo. series.

$$\begin{aligned} \Rightarrow \sum_{n=3}^{\infty} \frac{3^{n-1}}{5 \cdot 4^{n+3}} &= \frac{3^2}{5 \cdot 4^6} + \frac{3^3}{5 \cdot 4^7} + \frac{3^4}{5 \cdot 4^8} + \dots \\ &= \frac{3^2}{5 \cdot 4^6} \left[1 + \frac{3}{4} + \left(\frac{3}{4}\right)^2 + \dots \right] \\ &= \frac{3^2}{5 \cdot 4^6} \sum_{n=1}^{\infty} \left(\frac{3}{4}\right)^{n-1} \quad (*) \end{aligned}$$

Note: $S_n = a + ar + ar^2 + \dots + ar^{n-1}$

$$rS_n = ar + ar^2 + ar^3 + \dots + ar^n$$

$$S_n(1-r) = a - ar^n$$

$$\Rightarrow S_n = \frac{a - ar^n}{r-1} \quad \text{if } r \neq 1.$$

$$\Rightarrow \sum_{n=1}^{\infty} ar^{n-1} \left\{ \begin{array}{l} \text{conv to 0 if } a=0 \\ \text{conv to } \frac{a}{1-r} \text{ if } |r| < 1 \\ \text{div. otherwise.} \end{array} \right.$$

from (a) consider $\sum_{n=1}^{\infty} \left(\frac{3}{4}\right)^{n-1}$

\downarrow

$a=1$ and $r = \frac{3}{4} < 1$

$$\Rightarrow \sum_{n=1}^{\infty} \left(\frac{3}{4}\right)^{n-1} \text{ conv to } \frac{a}{1-r} = \frac{1}{1-\frac{3}{4}} = 4$$

$$\Rightarrow \sum_{n=1}^{\infty} a_n \text{ conv. to } 4 \cdot \frac{3^2}{5 \cdot 4^6} = \frac{9}{5 \cdot 4^5}$$

(b) $\sum_{n=1}^{\infty} \sin^{2n}(\theta)$ for $\theta \in (0, \pi/2)$

We have $\sum_{n=1}^{\infty} a_n$ where $a_n = (\sin^2(\theta))^n$

Te,

$$\begin{aligned} \sum_{n=1}^{\infty} a_n &= \sin^2(\theta) + (\sin^2(\theta))^2 + (\sin^2(\theta))^3 + (\sin^2(\theta))^4 + \dots \\ &= \sin^2(\theta) [1 + \sin^2(\theta) + \sin^4(\theta) + \sin^6(\theta) + \dots] \end{aligned}$$

$$= \sin^2(\theta) \underbrace{\sum_{n=1}^{\infty} (\sin^2(\theta))^{n-1}}_{\text{geo. series with } r = \sin^2(\theta) < 1 \text{ for } \theta \in (0, \pi/2)}, \quad \theta \in (0, \pi/2)$$

geo. series with $r = \sin^2(\theta) < 1$ for $\theta \in (0, \pi/2)$

$$a = 1$$

$$\Rightarrow \sum_{n=1}^{\infty} a_n \text{ conv. to } \sin^2(\theta) \cdot \frac{1}{1 - \sin^2 \theta} = \tan^2(\theta)$$

$$(c) \sum_{n=0}^{\infty} a_n \text{ where } a_n = \frac{2^{2n+1}}{\pi^{n-2}} = \frac{2 \cdot 4^n}{\pi^{n-2}}$$

$$\Rightarrow \sum_{n=0}^{\infty} a_n = 2 \cdot \pi^2 \underbrace{\sum_{n=0}^{\infty} \left(\frac{4}{\pi}\right)^n}_{\text{geo. series with } r = \frac{4}{\pi} > 1}$$

i.e., $\sum_{n=0}^{\infty} a_n$ is divergent

$\sum_{n=0}^{\infty} \left(\frac{4}{\pi}\right)^n$ divergent

$$(d) \text{ We have the series } \sum_{n=1}^{\infty} \frac{1}{n^{2+n}} = \sum_{n=1}^{\infty} \frac{1}{n} - \frac{1}{n+1}$$

↓
Telescoping series

let consider the n.th partial sum S_n of $\sum_{n=1}^{\infty} \frac{1}{n^{2+n}}$

We have $S_n = a_1 + a_2 + \dots + a_n$ where $a_n = \frac{1}{n^2+n}$

$$\Rightarrow S_n = 1 - \frac{1}{2} + \frac{1}{2} - \frac{1}{3} + \frac{1}{3} - \frac{1}{4} + \dots + \frac{1}{n} - \frac{1}{n+1}$$

$$\Rightarrow S_n = 1 - \frac{1}{n+1}$$

$$\sum_{n=1}^{\infty} a_n = \sum_{n=1}^{\infty} \frac{1}{n^2+n} = \lim_{n \rightarrow \infty} S_n = \lim_{n \rightarrow \infty} 1 - \frac{1}{n+1} = 1$$

If the limit exist.

Hence we have $\sum_{n=1}^{\infty} \frac{1}{n^2+n} = 1$.

(e) $\sum_{n=1}^{\infty} \frac{1}{n^2+2n}$

We have $\sum_{n=1}^{\infty} a_n$ where $a_n = \frac{1}{n^2+2n}$

Consider $a_n = \frac{1}{n(n+2)} = \frac{A}{n} + \frac{B}{n+2}$, find $A, B = ?$

$$A(n+2) + B(n) = 1$$

$$\text{If } n=0 \Rightarrow A = \frac{1}{2}$$

$$\text{If } n=-2 \Rightarrow B = -\frac{1}{2}$$

$$\Rightarrow \frac{1}{n(n+2)} = \frac{1}{2n} - \frac{1}{2n+4}$$

Consider the n -th partial sum S_n of $\sum_{n=1}^{\infty} a_n$

i.e.,

$$S_n = a_1 + a_2 + \dots + a_n$$

$$\begin{aligned}
 &= \frac{1}{2} - \cancel{\frac{1}{6}} \\
 &+ \frac{1}{4} - \cancel{\frac{1}{8}} \\
 &+ \cancel{\frac{1}{6}} - \cancel{\frac{1}{10}} \\
 &+ \cancel{\frac{1}{8}} - \cancel{\frac{1}{12}} \\
 &\quad \vdots \\
 &+ \cancel{\frac{1}{2(n-2)}} - \cancel{\frac{1}{2(n)}}
 \end{aligned}$$

$$+\frac{1}{2(n-1)} - \frac{1}{2(n+1)} \\ +\cancel{\frac{1}{2n}} - \frac{1}{2(n+2)}$$

$$\Rightarrow S_n = \frac{1}{2} + \frac{1}{4} - \frac{1}{2(n+1)} - \frac{1}{2(n+2)}$$

$$\Rightarrow \lim_{n \rightarrow \infty} S_n = \frac{1}{2} + \frac{1}{4} = \frac{3}{4} \quad (\in \mathbb{R}) \text{ exist}$$

So $\sum_{n=1}^{\infty} \frac{1}{n^2+2n} = \lim_{n \rightarrow \infty} S_n$ since the limit exist.

$$= \frac{3}{4} \quad (\text{conv to } \frac{3}{4})$$

3. Determine the convergence of the following series.

(a) $\sum_{n=2}^{\infty} \frac{2 \ln n^3 + 1}{5 \ln n^2 + 4},$

(b) $\sum_{n=0}^{\infty} \left(\frac{n+1}{n+2} \right)^n$

(c) $\sum_{n=10}^{\infty} \frac{1}{n^2 - 7n - 8},$

(d) $\sum_{n=1}^{\infty} \frac{2 + \sqrt{n}}{5n + 1},$

(e) $\sum_{n=1}^{\infty} \frac{4\pi^n + n^2}{e^{2n} + 1},$

(f) $\sum_{n=0}^{\infty} \frac{\sqrt[4]{2n^3 + n}}{\sqrt[3]{1 + 3n^5}},$

(g) $\sum_{n=1}^{\infty} \frac{\ln n}{\sqrt{n^5}}.$

(h) $\sum_{n=0}^{\infty} \frac{n+2}{n!},$

(i) $\sum_{n=1}^{\infty} \frac{n!}{2^{n^2}},$

(j) $\sum_{n=1}^{\infty} \frac{2^n}{n\sqrt{n}},$

(k) $\sum_{n=1}^{\infty} \frac{n!(n+1)!}{(3n)!}.$

Note: n-th term test.

If $\sum_{n=1}^{\infty} c_n$ is convergent then $\lim_{n \rightarrow \infty} c_n = 0$.

Soln: (a) We have $\sum_{n=2}^{\infty} a_n$ where $a_n = \frac{2\ln(n^3)+1}{5\ln(n^2)+4}$

Let consider the seq $\{a_n\}_{n=2,3,\dots}$

Let $f(x) = \frac{2\ln(x^3)+1}{5\ln(x^2)+4}$ be a func (diff on $(0, \infty)$)
st $f(n) = a_n \quad \forall n \geq 2$

$$\begin{aligned} \Rightarrow \lim_{x \rightarrow \infty} f(x) &= \lim_{x \rightarrow \infty} \frac{2\ln(x^3)+1}{5\ln(x^2)+4} \quad \left[\frac{\infty}{\infty} \right] \\ &\stackrel{\text{L'Ht}}{=} \lim_{x \rightarrow \infty} \frac{2 \cdot 3x^2/x^3}{5 \cdot 2x/x^2} = \lim_{x \rightarrow \infty} \frac{6}{x} \cdot \frac{x}{10} = \frac{3}{5} \end{aligned}$$

\Rightarrow Since $f(n) = a_n \quad \forall n \geq 2$, $\lim_{n \rightarrow \infty} a_n = \frac{3}{5} \neq 0$

Hence $\sum_{n=2}^{\infty} a_n$ is divergent by nth term test.

(b) We have $\sum_{n=h}^{\infty} a_n$ where $a_n = \left(\frac{n+1}{n+2}\right)^n$

Consider $\lim_{n \rightarrow \infty} \left(\frac{n+1}{n+2}\right)^n$

Let $f(n) = a_n$ where $f(x) = \left(\frac{x+1}{x+2}\right)^x$

find $\lim_{x \rightarrow \infty} \left(\frac{x+1}{x+2}\right)^x = ?$

Consider $\lim_{x \rightarrow \infty} x \cdot \ln\left(\frac{x+1}{x+2}\right) = \lim_{x \rightarrow \infty} \frac{\ln\left(\frac{x+1}{x+2}\right)}{1/x} \left[\frac{0}{0} \right]$

$$= \stackrel{\text{l'H}}{\lim_{x \rightarrow \infty}} \frac{x+2 - (x+1)}{(x+2)^2} \cdot \frac{x+2}{x+1} \cdot (-x^2)$$

$$= \lim_{x \rightarrow \infty} \frac{-x^2}{(x+2)(x+1)} = -1 \Rightarrow \lim_{x \rightarrow \infty} \left(\frac{x+1}{x+2}\right)^x = e^{-1} = \frac{1}{e} \neq 0$$

Since $f(n) = a_n$, $\lim_{n \rightarrow \infty} \left(\frac{n+1}{n+2}\right)^n = \frac{1}{e} \neq 0$, so by nth term test $\sum_{n=1}^{\infty} a_n$ is divergent.

(c) We have $\sum_{n=10}^{\infty} a_n$ where $a_n = \frac{1}{n^2 - 7n - 8}$

Note that $n^2 - 7n - 8 = (n+1)(n-8) = a_n > 0 \quad \forall n \geq 10$
 i.e., $a_n > 0 \quad \forall n \geq 10$.

condition of
LCT.

Note: LCT

Suppose $\{a_n\}$ & $\{b_n\}$ are positive sequences.

St $\lim_{n \rightarrow \infty} \frac{a_n}{b_n} = L$
 \downarrow
 exist or $\pm\infty$.

- if $L < \infty$ and $\sum_{n=1}^{\infty} b_n$ conv $\Rightarrow \sum_{n=1}^{\infty} a_n$ conv

- if $L > 0$ and $\sum_{n=1}^{\infty} b_n$ div $\Rightarrow \sum_{n=1}^{\infty} a_n$ div

Consider the series $\sum_{n=10}^{\infty} b_n$ where $b_n = \frac{1}{n^2} > 0 \quad \forall n \geq 10$

$$\Rightarrow \lim_{n \rightarrow \infty} \frac{a_n}{b_n} = \lim_{n \rightarrow \infty} \frac{1}{\frac{1}{n^2} - 7n - 8} \cdot n^2 = 1 = L < \infty$$

and $\sum_{n=10}^{\infty} \frac{1}{n^2}$ conv by p-series test ($p=2$)

Then $\sum_{10}^{\infty} a_n$ conv by LCT.

Note: p-series test

$$\sum_{n=1}^{\infty} n^{-p} = \begin{cases} \text{conv if } p > 1 \\ \text{div to } \infty \text{ if } p \leq 1 \end{cases}$$

(d) We have $\sum_{n=1}^{\infty} a_n$ where $a_n = \frac{2+\sqrt{n}}{5n+1} > 0 \quad \forall n \geq 1$.

Consider the series $\sum_{n=1}^{\infty} b_n$ where $b_n = \frac{1}{\sqrt{n}} > 0 \quad \forall n \geq 1$.

(we have positive sequences $\{a_n\}, \{b_n\}$, so we can use LCT)

$$\Rightarrow \lim_{n \rightarrow \infty} \frac{c_n}{b_n} = \lim_{n \rightarrow \infty} \frac{2 + \sqrt{n}}{\sqrt{n+1}} \cdot \sqrt{n} = \lim_{n \rightarrow \infty} \frac{2\sqrt{n} + n}{\sqrt{n+1}}$$

$$= \lim_{n \rightarrow \infty} \frac{\cancel{n} \left(\frac{2}{\sqrt{n}} + 1 \right)}{\cancel{n} \left(\frac{1}{\sqrt{n}} + \frac{1}{n} \right)} = \frac{1}{\frac{1}{2}} > 0$$

and $\sum_{n=1}^{\infty} \frac{1}{\sqrt{n}}$ is divergent by p-series test ($p=1/2$)

Hence $\sum_{n=1}^{\infty} c_n$ is divergent by LCT.

(e) We have $\sum_{n=1}^{\infty} a_n$ where $a_n = \frac{4\pi^n + n^2}{e^{2n} + 1} > 0 \quad \forall n \geq 1$

$$\text{We have } a_n = \frac{4\pi^n}{e^{2n} + 1} + \frac{n^2}{e^{2n} + 1}$$

$$\text{Let } b_n := \frac{4\pi^n}{e^{2n} + 1} > 0 \quad \forall n \geq 1 \quad \text{and} \quad b_n = \frac{4\pi^n}{e^{2n} + 1} \leq \frac{4\pi^n}{e^{2n}} = 4 \underbrace{\left(\frac{\pi}{e^2}\right)^n}_{:= c_n}$$

$$\text{Now we have } \sum_{n=1}^{\infty} c_n = \sum_{n=1}^{\infty} 4 \left(\frac{\pi}{e^2}\right)^n, \quad c_n > 0 \quad \forall n \geq 1$$

↳ geometric series

$$\text{with } r = \frac{\pi}{e^2} < 1$$

Since $r = \frac{\pi}{e^2} < 1$, $\sum_{n=1}^{\infty} 4 \left(\frac{\pi}{e^2}\right)^n$ is convergent

by comparison test, $\sum_{n=1}^{\infty} b_n$ also converges.

Now consider the series $\sum_{n=1}^{\infty} d_n$ where $d_n = \frac{n^2}{e^{2n} + 1}$

Note that $d_n = \frac{n^2}{e^{2n} + 1} > 0 \quad \forall n \geq 1$

and also $d_n = \frac{n^2}{e^{2n} + 1} \leq \frac{n^2}{e^{2n}} := f_n$

Now, let consider $\sum_{n=1}^{\infty} f_n = \sum_{n=1}^{\infty} \frac{n^2}{e^{2n}}$, where $f_n > 0 \quad \forall n \geq 1$

Let $f_n = f(n)$ where $f(x) = \frac{x^2}{e^{2x}}$ is positive, cont

and nonincreasing on $[1, \infty)$
 (exercise)

$$\Rightarrow \int_1^{\infty} f(x) dx = \int_1^{\infty} \frac{x^2}{e^{2x}} dx = \lim_{t \rightarrow \infty} \int_1^t \frac{x^2}{e^{2x}} dx$$

$$\begin{aligned} \text{Find } \int \frac{x^2}{e^{2x}} dx &= I \\ &\text{Let } u = x^2 \\ &du = 2x dx \\ &e^{-2x} dx = dv \\ &\frac{e^{-2x}}{-2} = v \end{aligned}$$

$$I = x^2 \underbrace{\frac{e^{-2x}}{-2}}_{-2} + \int \frac{e^{-2x}}{x} 2x dx$$

$$\hookrightarrow \text{IBP: } u = x \quad e^{-2x} dx = dv \\ du = dx \quad \frac{e^{-2x}}{-2} = v$$

$$\begin{aligned} I &= -\frac{x^2 e^{-2x}}{2} + \left[-x \frac{e^{-2x}}{2} + \int \frac{e^{-2x}}{2} dx \right] \\ &= -\frac{x}{2} e^{-2x} (x+1) + \frac{e^{-2x}}{2} \cdot \frac{1}{-2} + C \\ &= -\frac{(2x^2 + 2x + 1)}{4e^{2x}} + C \\ \Rightarrow \lim_{t \rightarrow \infty} \int_1^t \frac{x^2}{e^{2x}} dx &= \lim_{t \rightarrow \infty} -\frac{(2t^2 + 2t + 1)}{4e^{2t}} + \frac{5}{4e^2} \\ &= 0 + \frac{5}{4e^2} \end{aligned}$$

Hence, $\sum_{n=1}^{\infty} \frac{n^2}{e^{2n}}$ is conv by Integral test

\Rightarrow by CT, $\sum_{n=1}^{\infty} \frac{n^2}{e^{2n} + 1}$ is also conv.

$$\Rightarrow \sum_{n=1}^{\infty} \frac{u\pi^n + n^2}{e^{2n} + 1} = \underbrace{\sum_{n=1}^{\infty} \frac{u\pi^n}{e^{2n} + 1}}_{\text{conv}} + \underbrace{\sum_{n=1}^{\infty} \frac{n^2}{e^{2n} + 1}}_{\text{conv.}}$$

is also convergent.

Note: The Integral Test

Suppose $f(n)=a_n$, where f is positive, cont and non-increasing ω on an interval $[N, \infty)$ for some pos int N .
Then $\sum_{n=1}^{\infty} a_n$ and $\int_N^{\infty} f(t) dt$

either both converge or both diverge to infinity.

(f) We have $\sum_{n=0}^{\infty} a_n$ where $a_n = \frac{4\sqrt[4]{2n^3+n}}{3\sqrt[3]{1+3n^5}} > 0 \quad \forall n \geq 1$

Note that $\sum_{n=0}^{\infty} a_n = \sum_{n=1}^{\infty} a_n$ since $a_0 = 0$.

Consider the series $\sum_{n=1}^{\infty} b_n$ where

$$b_n = \frac{1}{n^{11/12}} > 0 \quad \forall n \geq 1$$

$$\begin{aligned} \Rightarrow \lim_{n \rightarrow \infty} \frac{a_n}{b_n} &= \lim_{n \rightarrow \infty} \frac{n^{3/4} \sqrt[4]{2 + 1/n^2}}{n^{5/3} \sqrt[3]{3 + 1/n^5}} \cdot n^{11/12} \\ &= \frac{2^{1/4}}{3^{1/5}} > 0 \end{aligned}$$

Note: $b_n = \frac{1}{n^{11/12}}$ and $\sum_{n=1}^{\infty} \frac{1}{n^{11/12}}$ divergent by p-series test ($p = \frac{11}{12} < 1$)

$\Rightarrow \sum_{n=1}^{\infty} a_n$ is divergent by LCT, ie, $\sum_{n=0}^{\infty} a_n$ diverges.

(g) We have $\sum_{n=1}^{\infty} a_n$ where $a_n = \frac{\ln(n)}{\sqrt{n^5}} > 0 \quad \forall n \geq 1$

We know that $\ln(x) < x \quad \forall x > 1$ (Proof: Exercise)

$$\Rightarrow 0 \leq a_n = \frac{\ln(n)}{n^{5/2}} \leq \frac{n}{n^{5/2}} = \frac{1}{n^{3/2}} := b_n$$

Consider the series $\sum_{n=0}^{\infty} b_n$ where $b_n = \frac{1}{n^{3/2}}$

since $p = \frac{3}{2} > 1$, $\sum_{n=0}^{\infty} b_n$ converges. by p-series test

Then $\sum_{n=0}^{\infty} a_n$ also conv by CT.

Soln:

Note: Ratio Test:

Let $\{a_n\}$ be seq st $a_n > 0$ and $\lim_{n \rightarrow \infty} \frac{a_{n+1}}{a_n} = p$

- if $0 \leq p < 1 \Rightarrow \sum a_n$ conv exist or no

- If $1 < p \leq \infty \Rightarrow \lim_{n \rightarrow \infty} a_n = \infty \wedge \sum a_n$ div.

(h) We have $a_n = \frac{n+2}{n!} > 0 \quad \forall n \geq 0$

$$\Rightarrow \lim_{n \rightarrow \infty} \frac{a_{n+1}}{a_n} = \lim_{n \rightarrow \infty} \frac{n+3}{(n+1)n!} \cdot \frac{n!}{n+2} = 0 \in [0, 1)$$

$\Rightarrow \sum_{n=0}^{\infty} a_n$ conv by RT

(i) We have $a_n = \frac{n!}{2^{n^2}} > 0 \quad \forall n \geq 1$

$$\Rightarrow \lim_{n \rightarrow \infty} \frac{a_{n+1}}{a_n} = \lim_{n \rightarrow \infty} \frac{(n+1)n!}{2^{(n+1)^2}} \cdot \frac{2^{n^2}}{n!} = \lim_{n \rightarrow \infty} \frac{n+1}{2^{2n+1}}$$

Let $f(n) = \frac{n+1}{2^{2n+1}}$ $\forall n \geq 1$ where $f(x) = \frac{x+1}{2^{2x+1}}$ $\begin{cases} \text{diff on} \\ (1, \infty) \end{cases}$

$$\text{Then } \lim_{x \rightarrow \infty} \frac{x+1}{2^{2x+1}} \left[\frac{\infty}{\infty} \right] \stackrel{\text{L'H}}{=} \lim_{x \rightarrow \infty} \frac{1}{2 \cdot \ln(2) \cdot (2^{2x+1})} = 0$$

$$\text{Since } f(n) = \frac{n+1}{2^{2n+1}} \quad \forall n \geq 1, \quad \lim_{n \rightarrow \infty} \frac{n+1}{2^{2n+1}} = 0$$

$$\text{Hence } \lim_{n \rightarrow \infty} \frac{a_{n+1}}{a_n} = 0 \in (0, 1)$$

ie, $\sum_{n=1}^{\infty} a_n$ conv by ratio test.

(J) We have $a_n = \frac{2^n}{n\sqrt{n}} > 0 \quad \forall n \geq 1$

$$\text{Consider } \lim_{n \rightarrow \infty} (a_n)^{1/n} = \lim_{n \rightarrow \infty} \frac{2}{n^{1/\sqrt{n}}}$$

$$\text{Note: } \lim_{x \rightarrow \infty} x^{1/\sqrt{x}} = ?$$

$$\lim_{x \rightarrow \infty} \frac{1}{\sqrt{x}} \cdot \ln(x) \left[\frac{\infty}{\infty} \right] \stackrel{\text{L'H}}{=} \lim_{x \rightarrow \infty} \frac{\frac{1}{x}}{\frac{1}{2\sqrt{x}}} = \lim_{x \rightarrow \infty} \frac{1}{x} \cdot 2\sqrt{x}$$

$$= \lim_{x \rightarrow \infty} \frac{2}{\sqrt{x}} = 0$$

$$\Rightarrow \lim_{x \rightarrow \infty} \ln \left(x^{1/x} \right) = 0 = \ln \left(\lim_{x \rightarrow \infty} x^{1/x} \right)$$

$$\Rightarrow \lim_{x \rightarrow \infty} x^{1/x} = e^0 = 1$$

Hence, $\lim_{n \rightarrow \infty} n^{1/n} = 1$ since $f(n) = n^{1/n} \forall n \geq 1$

$$\Rightarrow \lim_{n \rightarrow \infty} (a_n)^{1/n} = \lim_{n \rightarrow \infty} \frac{2}{n^{1/n}} = \frac{2}{1} = 2 > 1$$

Hence $\sum_{n=1}^{\infty} a_n$ is divergent by Root test.

(k) We have $a_n = \frac{n! (n+1)!}{(3n)!} > 0 \quad \forall n \geq 1$

$$\begin{aligned} \Rightarrow \lim_{n \rightarrow \infty} \frac{a_{n+1}}{a_n} &= \lim_{n \rightarrow \infty} \frac{(n+1)! (n+2)!}{(3n+3)!} \cdot \frac{(3n)!}{n! (n+1)!} \\ &= \lim_{n \rightarrow \infty} \frac{(n+2)(n+1) \cancel{n!} (3n+1)}{(3n+3)(3n+2)(3n+1) \cancel{(3n)!} \cancel{n!}} \\ &= 0 \in [0, 1] \end{aligned}$$

Hence $\sum_{n=1}^{\infty} a_n$ converges by Ratio test.

Note: The root test

Suppose $a_n > 0$ and $\lim_{n \rightarrow \infty} (a_n)^{1/n} = L$ exist or is ∞

- if $0 < L < 1$ then $\sum_{n=1}^{\infty} a_n$ conv

- if $1 < L \leq \infty \Rightarrow \lim_{n \rightarrow \infty} a_n = \infty$ and $\sum_{n=1}^{\infty} a_n$ diverges to ∞

- if $L=1 \Rightarrow \exists \text{ no information.}$