MATH120 2021-2 Recitation Problems - Week 01

1. For the given sequence determine whether it is monotone and whether it is convergent. If convergent, find its limit:

(a) $\{n - \sqrt{n+1}\sqrt{n+3}\}$

(b) $\left\{ \dfrac{(\ln n)^2}{n} \right\}$

(c) $\left\{ \left(1 + \dfrac{(-1)^n}{2}\right)^n \right\}$

(d) $\left\{ \dfrac{(n!)^2 2^n}{(2n)!} \right\}$

2. Find the limit of the sequence if it is convergent and explain why if it is divergent:

(a) $\left\{ \dfrac{\sin(2n)}{1+\sqrt{n}} \right\}$

(b) $\{\arctan(\ln n)\}$

(c) $\left\{ \dfrac{n!}{\pi^n} \right\}$

(d) $\left\{ \dfrac{(-4)^n n^{2n}}{(2n)! + n^n} \right\}$

3. Let the sequence $\{a_n\}$ be defined by $a_1 = 2$ and $a_{n+1} = \dfrac{a_n}{1+a_n}$ for $n \geq 1$.

(a) Show that $\{a_n\}$ is bounded.

(b) Show that $\{a_n\}$ is decreasing.

(c) Find $\lim\limits_{n\to\infty} a_n$.

Let $a_n = \frac{(-4)^n n^{2n}}{(2n)! + n^n} = \frac{(-1)^n \cdot 2^{2n} \cdot n^{2n}}{(2n)! + n^n}$

$$= \frac{(-1)^n (2n)^{2n}}{(2n)! + n^n} = \frac{(-1)^n (2n)^{2n}}{(2n)! \left(1 + \frac{n^n}{(2n)!}\right)}$$

consider the subsequence $\{a_{2n}\}$ ← even terms (positive terms)

$$a_{2n} = \frac{(4n)^{4n}}{(4n)! \left(1 + \frac{(2n)^{2n}}{(4n)!}\right)} = \frac{4n \cdot 4n \cdot 4n \cdot \cdots \cdot 4n}{4n(4n-1) \cdots 3 \cdot 2 \cdot 1} \cdot \frac{1}{\left(1 + \frac{(2n)^{2n}}{(4n)!}\right)}$$

$$\geq \frac{1}{1 + \frac{(2n)^{2n}}{(4n)!}}$$

$(*)\ \geq \frac{1}{2}$ since $\frac{2n^{2n}}{(4n)!} \geq 1$

Similarly consider the negative terms (odd terms)

$$-\frac{(2n)^{2n}}{(2n)! + n^n} \leq -\frac{1}{2} \quad \text{from } (*)$$

Claim: $\lim_{n \to \infty} a_n$ dne.

proof: (Proof by contradiction)

Suppose $\lim_{n\to\infty} a_n = L$ ← exist then $\lim_{n\to\infty} a_{2n} = \lim_{n\to\infty} a_{2n+1} = L$

but we have even terms $\geq \frac{1}{2}$, ie, $L \geq \frac{1}{2}$

and odd terms $\le -\frac{1}{2}$, ie $l \le -\frac{1}{2}$

so this is a contradiction,

ie, our assumption is wrong, ie, $\lim_{n \to \infty} a_n$ dne.

Math 120 Recitation

Week 1

1. For the given sequence determine whether it is monotone and whether it is convergent. If convergent, find its limit:

(a) $\{n - \sqrt{n+1}\sqrt{n+3}\}$

(b) $\left\{\dfrac{(\ln n)^2}{n}\right\}$

(c) $\left\{\left(1+\dfrac{(-1)^n}{2}\right)^n\right\}$

(d) $\left\{\dfrac{(n!)^2 2^n}{(2n)!}\right\}$

Soln: (a) Note: A seq $\{a_n\}$ be a func from $\mathbb{N}$ to $\mathbb{R}$, ie, $f : \mathbb{N} \to \mathbb{R}$

$$n \longmapsto f(n) = a_n$$

Let $f(x) = x - \sqrt{(x+1)(x+3)}$ where $f(n) = a_n \quad \forall n \geq 1$

diff. on $(0,\infty)$

Now consider $f'(x) = 1 - \dfrac{2x+4}{2\sqrt{x^2+4x+3}} = 1 - \dfrac{x+2}{\sqrt{x^2+4x+3}}$

Note: $\dfrac{x+2}{\sqrt{x+1}\,\sqrt{x+3}} = \sqrt{\dfrac{x^2+4x+4}{x^2+4x+3}} > 1 \quad \forall x \geq 1$

So $f'(x) = 1 - \dfrac{x+2}{\sqrt{x^2+4x+3}} < 0 \quad \forall x \geq 1$

$\Rightarrow$ $f(x)$ is a decreasing func on $(1,\infty)$

$\Rightarrow$ $f(n) = a_n$ is decreasing sequence, ie $\{a_n\}$ is monotone.

Let $a_n = n - \sqrt{(n+1)(n+3)}$, then consider $\lim_{n\to\infty} a_n$

$$\lim_{n\to\infty} a_n = \lim_{n\to\infty} \frac{n^2 - (n+1)(n+3)}{n + \sqrt{(n+1)(n+3)}}$$

$$= \lim_{n\to\infty} \frac{\cancel{n^2} - \cancel{n^2} - 4n - 3}{n + \sqrt{(n+1)(n+3)}} = \lim_{n\to\infty} \frac{-4n-3}{n+\sqrt{n^2+4n+3}}$$

$$= \lim_{n\to\infty} \frac{\cancel{n}(-4 - 3/n)}{\cancel{n}\left(1 + \sqrt{1 + \frac{4}{n} + \frac{3}{n^2}}\right)}$$

$$= \frac{-4}{2} = -2$$

$\Rightarrow$ $\{a_n\}$ is convergent sequence (conv to -2)

(b) Let $a_n = \frac{(\ln(n))^2}{n}$ for $n = 1, 2, 3, \dots$

$a_1 = 0$ and $a_n > 0$ $\forall n \geq 1$

Let $f(x) = \frac{(\ln(x))^2}{x}$ where $f(n) = a_n$ $\forall n \in \mathbb{Z}^+$ then (f diff on $(0,\infty)$)

$$f'(x) = \left[2\ln(x) \cdot \frac{1}{x} \cdot x - (\ln(x))^2\right] \frac{1}{x^2}$$

$$= \frac{\ln(x)(2 - \ln(x))}{x^2}$$

$\Rightarrow$ $f'(x) = 0$ if $\ln(x) = 0$ or $2 - \ln(x) = 0$

ie, $f'(x)=0$ if $x=1$ or $x=e^2$

	1		e^2	
f'		$+$		$-$
f		$\nearrow$		$\searrow$

f is increasing on $(1, e^2)$
f is dec. on (e^2, ∞)

so $\{a_n\}$ is increasing for $n=1,2,3,4,5,6,7$
and $\{a_n\}$ is decreasing for $n>7$

Thus, $\{a_n\}$ is <u>NOT</u> monotone.

Let $f(x) = \dfrac{(\ln(x))^2}{x} \Rightarrow f(n) = a_n$

consider $\displaystyle\lim_{x\to\infty} f(x) = \lim_{x\to\infty} \frac{\ln(x)^2}{x} \quad \left[\frac{\infty}{\infty}\right]$

$$\overset{L'H}{=} \lim_{x\to\infty} 2 \cdot \ln(x) \cdot \frac{1}{x} \quad \left[\frac{\infty}{\infty}\right]$$

$$\overset{L'H}{=} \lim_{x\to\infty} 2 \cdot \frac{1}{x}$$

$$= 0$$

since $\displaystyle\lim_{x\to\infty} f(x) = 0$, we have $\displaystyle\lim_{n\to\infty} a_n = 0$

ie, the seq $\{a_n\}$ is convergent (conv to 0.)

(c) Let $a_n = \left(1 + \frac{(-1)^n}{2}\right)^n$

we have $a_1 = \left(1 - \frac{1}{2}\right)^1 = \frac{1}{2}$

$a_2 = \left(1 + \frac{1}{2}\right)^2 = \left(\frac{3}{2}\right)^2 = \frac{9}{4}$

$a_3 = \left(1 - \frac{1}{2}\right)^3 = \frac{1}{8}$

so we have $a_1 < a_2$ but $a_2 > a_3$, so $\{a_n\}$ is NOT monotone.

consider $a_{2n} = \left(1 + \frac{(-1)^{2n}}{2}\right)^{2n} = \left(1 + \frac{1}{2}\right)^{2n} = \left(\frac{3}{2}\right)^{2n}$

and $a_{2n+1} = \left(1 + \frac{(-1)^{2n+1}}{2}\right)^{2n+1} = \left(1 - \frac{1}{2}\right)^{2n+1} = \left(\frac{1}{2}\right)^{2n+1}$

Note: If all subsequences of a_n conv. to the same limit value L ($\in \mathbb{R}$) then $\lim_{n\to\infty} a_n = L$

We have,

$\lim_{n\to\infty} a_{2n} = \lim_{n\to\infty} \left(\frac{3}{2}\right)^{2n} = \infty$

$\lim_{n\to\infty} a_{2n+1} = \lim_{n\to\infty} \left(\frac{1}{2}\right)^{2n+1} = 0$

Hence, $\lim_{n\to\infty} a_n$ dne., $\{a_n\}$ is divergent.

(d) let $a_n = \frac{(n!)^2 \, 2^n}{(2n)!}$ Note that $a_n > 0$ $\forall n \geq 1$.

Consider
$$\frac{a_{n+1}}{a_n} = \frac{[(n+1)!]^2 \, 2^{n+1}}{(2(n+1))!} \cdot \frac{(2n)!}{(n!)^2 \cdot 2^n}$$

$$= \frac{(n+1)^2 \cdot \cancel{(n!)^2} \cdot \cancel{2^n} \cdot 2}{(2n+2)\,(2n+1) \cdot \cancel{(2n)!}} \cdot \frac{\cancel{(2n)!}}{\cancel{(n!)^2} \cdot \cancel{2^n}}$$

$$= \frac{(n+1)^{\cancel{2}} \cdot \cancel{2}}{\cancel{2}\,\cancel{(n+1)} \cdot (2n+1)}$$

$$= \frac{n+1}{2n+1} < 1 \quad \text{since} \quad n+1 < 2n+1 \quad \forall n \geq 1$$

$\Rightarrow$ $a_{n+1} < a_n$ $\forall n \geq 1$

$\Rightarrow$ (a_n) is decreasing sequence. So it is monotone

Now consider
$$a_n = \frac{(n!)^2 \cdot 2^n}{(2n)!} = \frac{(n!)^{\cancel{2}} \cdot 2^n}{\cancel{n!}\,(n+1)(n+2)\cdots(2n)}$$

$$= \frac{1 \cdot 2 \cdot 3 \cdots n}{(n+1)(n+2)\cdots(2n)} \cdot 2 \cdot 2 \cdots 2$$

$$= \frac{1}{n+1} \cdot \underbrace{\frac{2}{n+2} \cdot \frac{3}{n+3} \cdots \frac{n}{2n}}_{\leq \frac{1}{2^{n-1}} \ (*)} \cdot 2^n$$

$$\leq \frac{1}{(n+1) \cdot 2^{n-1}} \cdot 2^n = \frac{2}{n+1}$$

Since $\lim_{n\to\infty} \frac{2}{n+1} = 0$ and $0 \leq c_n \leq \frac{2}{n+1}$

by Sequeeze thm $\lim_{n\to\infty} a_n = 0$

(*) must show $\frac{2}{n+2} \cdot \frac{3}{n+3} \cdots \frac{n}{n+n} \leq \frac{1}{2^{n-1}}$

Note that $\frac{2}{n+2} \cdot \frac{3}{n+3} \cdots \frac{n}{n+n} = \frac{(n!)^2 \cdot (n+1)}{(2n)!}$

$$= \frac{\cancel{n!} \cdot n! \cdot \cancel{(n+1)}}{\cancel{n!}\,\cancel{(n+1)} \cdot (n+2) \cdots (2n)}$$

$$= \frac{n!}{(n+2)(n+3)\cdots(n+n)}$$

$$= \frac{2 \cdot 3 \cdots n}{(n+2)(n+3)\cdots(n+n)}$$

$$:= c_n$$

ie, we'll show that $c_n \leq \frac{1}{2^{n-1}}$ $\forall n \geq 1$.

Use induction on n:

- if $n = 2 \Rightarrow c_2 = \frac{2}{4} \leq \frac{1}{2}$ ✓

- Suppose $c_n \leq \frac{1}{2^{n-1}}$, now show that $c_{n+1} \leq \frac{1}{2^n}$

Consider $c_{n+1} = \dfrac{[(n+1)!]^2 (n+2)}{(2(n+1))!} = \dfrac{(n+1)^2 \cdot (n!)^2 (n+2)}{(2n+2)(2n+1)(2n)!}$

$$= \frac{(n+1) \cdot (n!)^2}{(2n)!} \cdot \frac{(n+1) \cdot (n+2)}{(2n+2) \cdot (2n+1)}$$

$$= c_n \cdot \frac{n+2}{2(2n+1)}$$ Note : $n+2 \leq 2n+1$ $\forall n \geq 1$

$$\leq c_n \cdot \frac{1}{2} \quad \text{since} \quad \frac{n+2}{2n+1} \leq 1$$

$$\leq \frac{1}{2^{n-1}} \cdot \frac{1}{2} = \frac{1}{2^n}$$

Hence $c_n \leq \dfrac{1}{2^{n-1}}$

2. Find the limit of the sequence if it is convergent and explain why if it is divergent:

(a) $\left\{\dfrac{\sin(2n)}{1+\sqrt{n}}\right\}$

(b) $\{\arctan(\ln n)\}$

(c) $\left\{\dfrac{n!}{\pi^n}\right\}$

(d) $\left\{\dfrac{(-4)^n n^{2n}}{(2n)!+n^n}\right\}$

(a) Let $a_n = \dfrac{\sin(2n)}{1+\sqrt{n}}$ for $n = 1, 2, 3 \dots$

Now consider $|a_n| = \dfrac{|\sin(2n)|}{(1+\sqrt{n})}$ $(1+\sqrt{n}) > 0 \quad \forall n \geq 1$

$$0 \leq |a_n| \leq \frac{1}{1+\sqrt{n}}$$

Since $\lim\limits_{n\to\infty} \dfrac{1}{1+\sqrt{n}} = 0$, by Sequeeze thm

$$\lim_{n\to\infty} |a_n| = 0$$

NOTE: If $\lim\limits_{n\to\infty} |a_n| = 0$ then $\lim\limits_{n\to\infty} a_n = 0$

So we have $\lim\limits_{n\to\infty} a_n = 0 = \lim\limits_{n\to\infty} \dfrac{\sin(2n)}{1+\sqrt{n}}$

$\Downarrow$

$\{a_n\}$ is convergent.

(b) Let $a_n = \arctan(\ln n)$ and let $f(x) = \arctan(\ln x)$

then we have $f(n) = a_n$ for $n = 1, 2 \ldots$ (f diff $(1, \infty)$)

arctan x cont

We have $\lim_{x \to \infty} \ln(x) = \infty$,

so $\lim_{n \to \infty} \arctan(\ln(x)) = \arctan\left(\lim_{x \to \infty} \ln(x)\right) = \pi/2$

Hence, we have $\lim_{n \to \infty} a_n = \frac{\pi}{2}$, so $\{a_n\}$ is conv.

(c) Let $a_n = \frac{n!}{\pi^n}$ for $n = 1, 2 \ldots$

Consider $a_n = \frac{n!}{\pi^n} = \frac{1 \cdot 2 \cdot 3 \cdot 4 \cdot \ldots \cdot n}{\pi \cdot \pi \cdot \pi \cdot \ldots \cdot \pi}$

$$\geq \frac{4}{\pi} \frac{4}{\pi} \cdot \frac{4}{\pi} \cdots \frac{4}{\pi} \cdot \frac{3 \cdot 2 \cdot 1}{\pi \, \pi \, \pi}$$

$$\geq \left(\frac{4}{\pi}\right)^{n-3} \cdot \frac{6}{\pi^3} := b_n$$

We have $\lim_{n \to \infty} \left(\frac{4}{\pi}\right)^{n-3} \cdot \frac{6}{\pi^3} = \infty = \lim_{n \to \infty} b_n$

$\Downarrow$

$\{b_n\}$ is unbounded

($\nexists$ no upper bound)

Since $0 < a_n$ and $a_n \geq b_n$ and $\{b_n\}$ is unbdd,

$\{a_n\}$ is also unbounded.

Now, we'll show that $\{a_n\}$ is increasing for $n \geq 3$

let consider $\frac{a_{n+1}}{a_n} = \frac{(n+1)!}{\pi^{n+1}} \cdot \frac{\pi^n}{n!} = \frac{n+1}{\pi} > 1 \quad \forall n \geq 3$

ie, $a_{n+1} \geq a_n \quad \forall n \geq 3 \Rightarrow \{a_n\}$ is increasing for $n \geq 3$

Since $a_n > 0 \quad \forall n \geq 1$ and $\{a_n\}$ increasing $\forall n \geq 3$ and $\{a_n\}$ is unbdd we have $\lim_{n \to \infty} a_n = \infty$

ie, $\{a_n\}$ is divergent

(d) let $a_n = \frac{(-4)^n \, n^{2n}}{(2n)! + n^n} = \frac{(-1)^n \cdot 2^{2n} \cdot n^{2n}}{(2n)! + n^n}$

$$= \frac{(-1)^n (2n)^{2n}}{(2n)! + n^n} = \frac{(-1)^n (2n)^{2n}}{(2n)! \left(1 + \frac{n^n}{(2n)!}\right)}$$

consider the subsequence $\{a_{2n}\}$ even terms (positive terms)

$$a_{2n} = \frac{(4n)^{4n}}{(4n)! \left(1 + \frac{(2n)^{2n}}{(4n)!}\right)} = \frac{4n \cdot 4n \cdot 4n \cdots \cdot 4n}{4n(4n-1) \cdots 3 \cdot 2 \cdot 1} \cdot \frac{1}{\left(1 + \frac{(2n)^{2n}}{(4n)!}\right)}$$

$$\geq \frac{1}{1 + \frac{(2n)^{2n}}{(4n)!}}$$

$$(*) \geq \frac{1}{2} \quad \text{since} \quad \frac{2n^{2n}}{(4n)!} \geq 1$$

Similarly consider the negative terms (odd terms)

$$-\frac{(2n)^{2n}}{(2n)!+n^n} \le \frac{-1}{2} \quad \text{from } (*)$$

<u>Claim:</u> $\lim_{n\to\infty} a_n$ dne.

<u>proof:</u> (Proof by contradiction)

Suppose $\lim_{n\to\infty} a_n = L$ ← exist then $\lim_{n\to\infty} a_{2n} = \lim_{n\to\infty} a_{2n+1} = L$

but we have even terms $\geq \frac{1}{2}$, ie, $L \geq \frac{1}{2}$

and odd terms $\leq \frac{-1}{2}$, ie $L \leq \frac{-1}{2}$

so this is a contradiction,

ie, our assumption is wrong, ie, $\lim_{n\to\infty} a_n$ dne.

3. Let the sequence $\{a_n\}$ be defined by $a_1 = 2$ and $a_{n+1} = \dfrac{a_n}{1+a_n}$ for $n \geq 1$.

(a) Show that $\{a_n\}$ is bounded.

(b) Show that $\{a_n\}$ is decreasing.

(c) Find $\lim_{n\to\infty} a_n$.

Soln: (a) $a_1 = 2$

$$a_{n+1} = \frac{a_n}{1+a_n}$$

We have $a_2 = \dfrac{a_1}{1+a_1} = \dfrac{2}{1+2} = \dfrac{2}{3}$

$$a_3 = \frac{a_2}{1+a_2} = \frac{2/3}{1+2/3} = \frac{2}{3} \cdot \frac{3}{5} = \frac{2}{5}$$

$\vdots$

Claim: $a_n > 0 \quad \forall n \geq 1$

proof: use induction on n.

for $n=1$, $a_1 = 2 > 0$

assume $a_n > 0$ then must show $a_{n+1} > 0$

but we have $a_{n+1} = \dfrac{\overset{>0}{a_n}}{\underset{>0}{1+a_n}} > 0$

Hence, $a_n > 0 \quad \forall n \geq 1$

claim: $a_n \leq 2 \quad \forall n \geq 1$

Induction on n.

if $n=1$, $a_1 = 2 \leq 2$

Assume $a_n \leq 2$ then show that $a_{n+1} \leq 2$

We have $a_{n+1} = \underset{>0}{\frac{a_n}{1+a_n}} \leq \frac{a_n}{1} \leq 2$

$\Downarrow$

$a_{n+1} \leq 2$

Hence $0 \leq a_n \leq 2$ $\forall n = 1, 2, \ldots$, ie, $\{a_n\}$ is bounded

(b) Show $\{a_n\}$ is decreasing.

Consider $a_{n+1} = \underset{>0}{\frac{a_n}{1+a_n}} \leq \frac{a_n}{1} = a_n$

$\Downarrow$

$a_{n+1} \leq a_n$ $\forall n = 1, 2 \ldots$

$\Downarrow$

$\{a_n\}$ is decreasing

(c) $\lim_{n\to\infty} a_n = ?$

By part (a), $\{a_n\}$ is bdd and by part (b) $\{a_n\}$ is decreasing so $\{a_n\}$ is convergent, ie $\lim_{n\to\infty} a_n =$ exist. let say $\lim_{n\to\infty} a_n = L$

$\Downarrow$

$\lim_{n\to\infty} a_{n+1} = L$

Note:

Since $0 \le a_n \le 2 \quad \forall n \geq 1$,

$$0 \le \lim_{n\to\infty} a_n = L \le 2$$

since we have $a_{n+1} = \dfrac{a_n}{1+a_n}$

$$\lim_{n\to\infty} a_{n+1} = L = \lim_{n\to\infty} \frac{a_n}{1+a_n} = \frac{\lim_{n\to\infty} a_n}{\underbrace{1+\lim_{n\to\infty} a_n}_{\neq 0}} = \frac{L}{1+L}$$

$$\Rightarrow \quad L = \frac{L}{1+L}$$

$$\Rightarrow \quad L^2 + L - L = 0$$

$$\Rightarrow \quad L = 0$$

ie, $\lim_{n\to\infty} a_n = L = 0.$