Q4

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4. Write the following limits as definite integrals

a) $\lim_{n\to\infty}\sum_{i=1}^{n}\left(2\left(\frac{i}{n}\right)^2+\frac{i}{n}\right)\left(\frac{1}{n}\right)$

b) $\lim_{n\to\infty}\sum_{i=1}^{n}\frac{i-1+2n}{i-1+n}\left(\frac{1}{n}\right)$

c) $\lim_{n\to\infty}\sum_{i=1}^{n}\frac{9i+3n}{3in+2n^2}$

d) $\lim_{n\to\infty}\sum_{i=1}^{n}\left(5+\frac{3i}{n}\right)^4\left(\frac{2}{n}\right)$ Exercise

$\sum_{i=1}^{n} f(c_{i-1})\Delta x_i$

$\sum_{i=1}^{n} f(c_i)\Delta x_i$

$c_0 \quad c_1 \quad c_2 \quad \ldots \quad c_n$

Soln:

a) Choose the sampling point $c_i = \frac{i}{n}$ for $i = 1, 2, 3, \ldots, n$

Then $c_i = \frac{i}{n}$ represents the right endpoints of the subdivisions of the interval of n equal sized $[0,1]$ and $\Delta x_i = \frac{1}{n}$ (Remember: $c_i = 0 + \frac{1}{n}\cdot i$)

$[a,b] \to [0,1]$, $\frac{b-a}{n} = \frac{1}{n}$

Thus $\lim_{n\to\infty}\sum_{i=1}^{n}\left(2\left(\frac{i}{n}\right)^2+\frac{i}{n}\right)\frac{1}{n} = \lim_{n\to\infty}\sum_{i=1}^{n}(2c_i^2+c_i)\cdot\Delta x_i$

let $f(x) = 2x^2+x$ $\quad = \lim_{n\to\infty}\sum_{i=1}^{n} f(c_i)\cdot\Delta x_i$

$= \int_0^1 f(x)\,dx = \int_0^1 (2x^2+x)\,dx$

b) $\lim_{n\to\infty}\sum_{i=1}^{n}\frac{i-1+2n}{i-1+n}\left(\frac{1}{n}\right)$

$= \lim_{n\to\infty}\sum_{i=1}^{n}\frac{(i-1+2n)\cdot\frac{1}{n}}{(i-1+n)\cdot\frac{1}{n}}\cdot\left(\frac{1}{n}\right) = \lim_{n\to\infty}\sum_{i=1}^{n}\frac{\frac{i-1}{n}+\frac{2\cancel{n}}{\cancel{n}}}{\frac{i-1}{n}+\frac{\cancel{n}}{\cancel{n}}}\cdot\left(\frac{1}{n}\right)$

$= \lim_{n\to\infty}\sum_{i=1}^{n}\frac{\frac{i-1}{n}+2}{\frac{i-1}{n}+1}\left(\frac{1}{n}\right)$ $\to \frac{1}{n} = \frac{b-a}{n} = \frac{1-0}{n}$

$0 \qquad\qquad 1$

$c_i = \frac{i-1}{n}$ (Remember $c_i = 0 + \frac{1}{n}\cdot(i-1)$)

Now, choose the sampling pt.

Then c_i represents left endpoints of n equal sized subdivisions of the interval $[0,1]$ and $\Delta x_i = 1/n$ for $i = 1, 2, \ldots, n$

Thus, $\lim_{n\to\infty}\sum_{i=1}^{n}\frac{\frac{i-1}{n}+2}{\frac{i-1}{n}+1}\left(\frac{1}{n}\right) = \lim_{n\to\infty}\sum_{i=1}^{n}\frac{c_i+2}{c_i+1}\Delta x_i$

let $f(x) = \frac{x+2}{x+1}$ $\quad = \lim_{n\to\infty}\sum_{i=1}^{n} f(c_i)\Delta x_i$

$= \int_0^1 f(x)\,dx = \int_0^1 \frac{x+2}{x+1}\,dx$

c) $\lim \sum^{n} \frac{9i+3n}{\ldots} = \lim \sum \frac{9i+3n}{\ldots}$

(c) $\lim_{n\to\infty} \sum_{i=1}^{n} \frac{9i+3n}{3in+2n^2} = \lim_{n\to\infty} \sum_{i=1}^{n} \frac{9i+3n}{n(3i+2n)}$

$c_i = \frac{i}{n}$ $[0,1]$

$c_i = 0 + \underbrace{\Delta x}_{\frac{1}{n}} \cdot i$

$= \lim_{n\to\infty} \sum_{i=1}^{n} \frac{9i+3n}{3i+2n} \underbrace{\left(\frac{1}{n}\right)}_{\Delta x} = \lim_{n\to\infty} \sum_{i=1}^{n} \frac{(9i+3n)\frac{1}{n}}{(3i+2n)\frac{1}{n}} \left(\frac{1}{n}\right)$

$= \lim_{n\to\infty} \sum_{i=1}^{n} \frac{9\left(\frac{i}{n}\right)+3}{3\left(\frac{i}{n}\right)+2} \left(\frac{1}{n}\right)$

$c_i = \frac{i}{n}$ for $i = 1, \dots, n$. Then c_i represents right endpoint of n equal-sized subdivisions of the interval $\underline{[0,1]}$ and $\Delta x_i = \frac{1}{n}$ for $i = 1, \dots, n$

$\lim_{n\to\infty} \sum_{i=1}^{n} \frac{9\left(\frac{i}{n}\right)+3}{3\left(\frac{i}{n}\right)+2} \left(\frac{1}{n}\right) = \lim_{n\to\infty} \sum_{i=1}^{n} \frac{9c_i+3}{3c_i+2} \Delta x_i = \lim_{n\to\infty} \sum_{i=1}^{n} f(c_i) \Delta x_i$

$\left(\text{Let } f(x) = \frac{9x+3}{3x+2} \right)$ $= \int_0^1 \frac{9x+3}{3x+2}\, dx$

If we choose $\boxed{c_i = \frac{3i}{n}}$, then interval will be $[0,3]$ and $\boxed{\Delta x_i = \frac{3}{n}}$ $\left(c_i = 0 + \frac{3}{n} \cdot i\right)$

Thus $\lim_{n\to\infty} \sum_{i=1}^{n} \frac{3}{n} \frac{(3i+n)}{(3i+2n)} = \lim_{n\to\infty} \sum_{i=1}^{n} \left(\frac{3}{n}\right) \frac{(3i+n)\frac{1}{n}}{(3i+2n)\frac{1}{n}}$

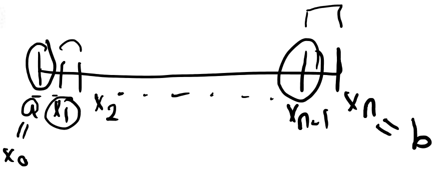
$= \lim_{n\to\infty} \sum_{i=1}^{n} \left(\frac{3}{n}\right) \frac{\frac{3i}{n}+1}{\frac{3i}{n}+2} = \lim_{n\to\infty} \sum_{i=1}^{n} f(c_i) \cdot \Delta x_i = \int_0^3 \frac{x+1}{x+2}\, dx$

let $f(x) = \frac{x+1}{x+2}$

1. If f is continuous and nondecreasing on $[a, b]$, and P_n is the partition of $[a, b]$ into n subintervals of equal length, show that

$$U(f, P_n) - L(f, P_n) = \frac{(b-a)(f(b) - f(a))}{n}$$

and by using this equality prove that f must be integrable on $[a, b]$.



Soln: let $\Delta x = \frac{b-a}{n}$ and $x_i = a + \Delta x \cdot i$ where $1 \leq i \leq n$.

Since f is cont. and nondecreasing

$$L(f, P_n) = \Delta x \cdot \left(f(a) + f(x_1) + f(x_2) + \dots + f(x_{n-1})\right) = \frac{b-a}{n}\left(f(a) + \sum_{i=1}^{n-1} f(x_i)\right)$$

$$U(f, P_n) = \Delta x \cdot \left(f(x_1) + f(x_2) + \dots + f(x_{n-1}) + f(b)\right) = \frac{b-a}{n}\left(\sum_{i=1}^{n-1} f(x_i) + f(b)\right)$$

$$U(f, P_n) - L(f, P_n) = \frac{b-a}{n} \cdot \left[\cancel{\sum_{i=1}^{n-1} f(x_i)} + f(b) - f(a) - \cancel{\sum_{i=1}^{n-1} f(x_i)}\right]$$

$$= \frac{b-a}{n} \cdot \left(f(b) - f(a)\right)$$

Since $\lim_{n \to \infty} \left(U(f, P_n) - L(f, P_n)\right) = \lim_{n \to \infty} \frac{(b-a)(f(b)-f(a))}{n} = 0$,

f must be integrable on $[a, b]$.

$$L(f, P_n) \leq \int_a^b f(x)\,dx \leq U(f, P_n)$$

Q2

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2. Using the property of integrals, verify the following inequalities

a) $\frac{\pi\sqrt{2}}{24} \le \int_{\pi/6}^{\pi/4} \cos x\,dx \le \frac{\pi\sqrt{3}}{24}$

b) $\frac{13}{10} \le \int_{-1}^{1} \frac{1}{1+x^2}\,dx \le \frac{18}{10}$

Soln : a) $\left[\frac{\pi}{6}, \frac{\pi}{4}\right]$

$x_0 = \frac{\pi}{6}$ $\quad x_1 = \frac{\pi}{4}$ $\quad \boxed{n=1}$

$x_n - x_0$

$\Delta x = \frac{b-a}{n} = \frac{x_1 - x_0}{1} = \pi/4 - \pi/6 = \frac{\pi}{12}$

$$\mathcal{U}(f,P) = \max_{x\in\left[\frac{\pi}{6},\frac{\pi}{4}\right]} \{\cos x\} \cdot \frac{\pi}{12} = \underbrace{\cos\left(\frac{\pi}{6}\right)}_{\frac{\sqrt{3}}{2}} \cdot \frac{\pi}{12} = \frac{\pi\sqrt{3}}{24}$$

$$\mathcal{L}(f,P) = \min_{x\in\left[\frac{\pi}{6},\frac{\pi}{4}\right]} \{\cos x\} \cdot \frac{\pi}{12} = \underbrace{\cos\left(\frac{\pi}{4}\right)}_{\frac{\sqrt{2}}{2}} \cdot \frac{\pi}{12} = \frac{\pi\sqrt{2}}{24}$$

$$\frac{\pi\sqrt{2}}{24} = \mathcal{L}(f,P) \le \int_{\pi/6}^{\pi/4} \cos x\,dx \le \mathcal{U}(f,P) = \frac{\pi\sqrt{3}}{24}$$ ✓

Try :

b) $-1 \quad 0 \quad 1$

$n=2$, $\Delta x = \frac{1-(-1)}{2} = 1$

$$\mathcal{U}(f,P) = \sum_{i=1}^{2} \max_{x\in[x_{i-1},x_i]} \left\{\frac{1}{1-x^2}\right\} \cdot \Delta x = (1+1)\cdot 1 = 2$$

$-1 \quad -1/2 \quad 0 \quad 1/2 \quad 1$

$\boxed{n=4} \Rightarrow \Delta x = \frac{1-(-1)}{4} = \frac{1}{2}$

$$\mathcal{U}(f,P) = \sum_{i=1}^{4} \max_{x\in[x_{i-1},x_i]} \left\{\frac{1}{1-x^2}\right\} \cdot \Delta x = \left(f\left(-\tfrac{1}{2}\right) + f(0) + f(0) + f(1/2)\right)\cdot 1/2 = \frac{18}{10}$$

$$\mathcal{L}(f,P) = \sum_{i=1}^{4} \min_{x\in[x_{i-1},x_i]} \left\{\frac{1}{1-x^2}\right\} \cdot \Delta x = \left(f(-1) + f\left(-\tfrac{1}{2}\right) + f\left(\tfrac{1}{2}\right) + f(1)\right)\cdot \frac{1}{2} = \frac{13}{10}$$

$$\frac{13}{10} = \mathcal{L}(f,P) \le \int_{-1}^{1} \frac{1}{1-x^2}\,dx \le \mathcal{U}(f,P) = \frac{18}{10}$$ ✓

Q3

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3. Approximate the following integrals using both the Trapezoid rule and Simpson's rule.

a) $\int_0^3 \sqrt{4+x^3}\,dx$, $n=3$

b) $\int_0^1 \sin^2(\pi x)\,dx$, for 6 sub-intervals

0 1 2 3

Recall: • $T_n = \frac{b-a}{n}\left(\frac{f(x_0)}{2} + f(x_1) + f(x_2) + \dots f(x_{n-1}) + \frac{f(x_n)}{2}\right)$ ($x_0 = a$, $x_n = b$)

• $S_{2n} = \frac{b-a}{3.(2n)} \cdot \left(f(a) + 4\sum_{i=1}^{n} f(x_{2i-1}) + 2\sum_{i=1}^{n-1} f(x_{2i}) + f(b)\right)$

Solution: a) $\Delta x = \frac{3-0}{3} = 1$

$T_n = 1.\left(\frac{f(0)}{2} + f(1) + f(2) + \frac{f(3)}{2}\right) = 1.\left(\frac{2}{2} + \sqrt{5} + \sqrt{12} + \frac{\sqrt{31}}{2}\right) \cong 10{,}48$ ($\sqrt{12} = 2\sqrt{3}$)

• $n=3 \Rightarrow 2n=6$, so we have 6 subintervals

x_0 x_1 x_2 x_3 x_4 x_5 x_6

0 1/2 1 3/2 2 5/2 3

$S_6 = \frac{(3-0)}{3.2.3}\left(f(0) + 4\sum_{i=1}^{3} f(x_{2i-1}) + 2\sum_{i=1}^{2} f(x_{2i}) + f(3)\right)$

$f(x) = \sqrt{4+x^3}$

$= \frac{1}{6}\cdot\left(2 + 4.\left(f(x_1)+f(x_3)+f(x_5)\right) + 2\left(f(x_2)+f(x_4)\right) + \sqrt{31}\right)$

$= \frac{1}{6}\left(2 + 4\left(\frac{\sqrt{33}}{\sqrt{8}} + \frac{\sqrt{59}}{\sqrt{8}} + \frac{\sqrt{157}}{\sqrt{8}}\right) + 2\left(\sqrt{5} + 2\sqrt{3}\right) + \sqrt{31}\right)$

b) $\int_0^1 \sin^2(\pi x)\,dx$ for 6-subinterval → Trapezoid $n=6$, Simpson's $n=3$

$T_6 = \frac{1}{6}\cdot\left(\frac{f(0)}{2} + f\left(\frac{1}{6}\right) + f\left(\frac{2}{6}\right) + f\left(\frac{3}{6}\right) + f\left(\frac{4}{6}\right) + f\left(\frac{5}{6}\right) + \frac{f(1)}{2}\right)$

0 1/6 2/6 3/6 4/6 5/6 1

x_0 x_1 x_2 x_3 x_4 x_5 x_6

$= \frac{1}{6}\cdot\left(0 + \left(\frac{1}{2}\right)^2 + \left(\frac{\sqrt{3}}{2}\right)^2 + 1 + \left(\frac{\sqrt{3}}{2}\right)^2 + \left(\frac{1}{2}\right)^2 + 0\right)$

$= \frac{1}{6}\cdot\left(\underbrace{\frac{1}{4} + \frac{3}{4}}_{=1} + 1 + \underbrace{\frac{3}{4} + \frac{1}{4}}_{=1}\right) = \frac{3}{6} = 1/2$ //

$S_6 = \frac{1}{3.2.3}\left(f(0) + 4\sum_{i=1}^{3} f(x_{2i-1}) + 2\sum_{i=1}^{2} f(x_{2i}) + f(1)\right)$

$= \frac{1}{18}\cdot\left(0 + 4\left(f\left(\frac{1}{6}\right) + f\left(\frac{3}{6}\right) + f\left(\frac{5}{6}\right)\right) + 2\left(f\left(\frac{2}{6}\right) + f\left(\frac{4}{6}\right)\right) + 0\right)$

$$= \frac{1}{18} \cdot \left(0 + 4\left(f\left(\frac{1}{6}\right) + f\left(\frac{3}{6}\right) + f\left(\frac{5}{6}\right) \right) + 2\left(f\left(\frac{2}{6}\right) + f\left(\frac{4}{6}\right) \right) + 0 \right)$$

$$= \frac{1}{18} \cdot \left(0 + 4\left(\frac{1}{4} + 1 + \frac{1}{4} \right) + 2\left(\frac{3}{4} + \frac{3}{4} \right) + 0 \right)$$

$$= \frac{1}{18} \cdot \left(1 + 4 + 1 + 3 \right) = \frac{1}{18} \cdot 9 = \frac{1}{2} \,//$$

Q4

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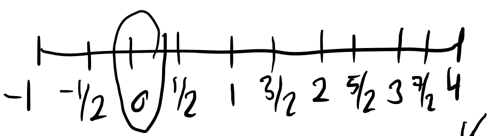
4. In estimating $\int_{-1}^{4} \cos(4x)\,dx$ using the Trapezoid Rule and the Simpson's Rule with 10 subintervals, find the error bounds.

Recall: • Error estimate for the Trapezoid Rule: If f has a continuous 2nd derivative on $[a,b]$ and $K := \max\{|f''(x)| \mid x \in [a,b]\}$,

then $$\left| \int_a^b f(x)\,dx - T_n \right| \le \frac{K\cdot(b-a)^3}{12n^2}$$

$$\left| \int_{-1}^{4} \cos(4x)\,dx - T_{10} \right| \le \frac{K\cdot(4-(-1))^3}{12\cdot(10)^2} \Rightarrow K = ?$$

For K: $f(x) = \cos(4x)$, $f'(x) = -\sin(4x)\cdot 4$, $|f''(x)| = |-\cos(4x)\cdot 4\cdot 4|$
$= |-16\cos(4x)|$



$\Delta x = \frac{5}{10} = 1/2$

$K = |-16| = 16$

$$\left| \int_{-1}^{4} \cos(4x)\,dx - T_{10} \right| \le \frac{\cancel{16}^{4}\cdot \cancel{125}^{5}}{_{3}\cancel{12}\cdot \cancel{100}_{4}} = \frac{5}{3}$$

Recall: • Error estimate for The Simpson's Rule: If f has a continuous 4th derivative on $[a,b]$ s.t. $\max\{|f^{(4)}(x)| \mid x \in [a,b]\} = K$

$n=3$!

S_{2n} subinterval 6

S_n — n even

$$\left| \int_a^b f(x)\,dx - S_n \right| \le \frac{K\cdot(b-a)^5}{180n^4}$$

n should be even

$n = 2k$, 10, 5

$f''(x) = -16\cos(4x)$, $f'''(x) = 16\sin(4x)4 = 64\sin(4x)$, $f^{IV}(x) = 256\cos(4x)$

at $x = 0 \Rightarrow K = 256$

$$\left| \int_{-1}^{4} \cos(4x)\,dx - S_{10} \right| \le \frac{256\cdot 5^5}{180\cdot 10^4} = \frac{256\cdot \cancel{5^5}}{\cancel{180}_{36}\cdot \cancel{5^4}\cdot \cancel{2^4}} = \frac{\cancel{16}^{4}}{\cancel{36}_{9}} = \frac{4}{9}$$

S_{2n} !

5. How large should n be for each to guarantee that the Trapezoid Rule and the Simpson Rule approximation to $\int_{-2}^{1} -x^4 - 2x^3 + 12x^2 + 2x - 5\,dx$ are accurate within 0.1?

$|\text{Error}| \leq 0.1$

Soln:

$\text{Error for } T_n = \frac{K.(b-a)^3}{12n^2} \leq 0.1$

$\frac{\cancel{27}^{9} . 3^3}{\cancel{12}_{4} . n^2} \leq \frac{1}{10}$

$\frac{243}{4n^2} \leq \frac{1}{10}$

$2430 \leq 4n^2$

$\sqrt{607.5} \leq \sqrt{n^2}$

$24.65 \leq n = \underline{\underline{25}}$

For T_n: $K = \max\{|f''(x)| \mid x \in [a,b]\}$

$f(x) = -x^4 - 2x^3 + 12x^2 + 2x - 5$

$f'(x) = -4x^3 - 6x^2 + 24x + 2$

$\Rightarrow f''(x) = -12x^2 - 12x + 24$

$\hookrightarrow f'''(x) = -24x - 12 = 0$

$\Leftrightarrow x = -1/2$, endpoints: $1, -2$

$K = \boxed{f''(-1/2) = 27}$, $f''(1) = 0$

$f''(-2) = 0$

For Simpson: $f^{IV}(x) = -24$

$K = |-24| = \underline{\underline{24}}$

$\frac{K.(b-a)^5}{180\,n^4} \leq \frac{1}{10}$

$\Rightarrow \frac{\cancel{24}^{2} . 3^5}{_{15}\cancel{180}\, n^4} \leq \frac{1}{10} \Rightarrow \cancel{20}_{4} . \cancel{3^5}_{3^4} \leq \cancel{15}_{\cancel{3}} n^4$

$\Rightarrow \sqrt[4]{4 . 3^4} \leq \sqrt[4]{n^4} \Rightarrow n \geq 4.24$

should be even so $n \neq 5$

Choose, $n = 6$!

Q6

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6. Given $\int_0^3 2x^2 - 1\,dx$ find c such that $f(c)$ equals the average value of $f(x) = 2x^2 - 1$ over $[0, 3]$.

7. Find the point c for a function $f(x) = 3x^2 - 2x$ that satisfies the Mean Value Theorem for integrals on the interval $[1, 4]$.