Q1

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Q1: Rewrite the following expressions by using summation notation

a) $20+25+30+35+\dots+100=?$

b) $4(0.1)+4.(0.01)+4.(0.001)+\dots=?$

c) $\frac{1}{8}+\frac{1}{16}+\frac{1}{32}+\dots$ starting at $i=3$ & $i=0$

Soln: a) $4.5+5.5+6.5+7.5+\dots+20.5=\sum_{i=4}^{20}5i$

b) $4.\left(\frac{1}{10}\right)+4.\left(\frac{1}{\underbrace{100}_{10^2}}\right)+4.\left(\frac{1}{\underbrace{1000}_{10^3}}\right)+\dots=\sum_{i=1}^{\infty}4\frac{1}{10^i}$

c) $\frac{1}{2^3}+\frac{1}{2^4}+\frac{1}{2^5}+\dots=\sum_{i=3}^{\infty}\frac{1}{2^{\boxed{i}}}=\sum_{i=0}^{\infty}\frac{1}{2^{i+3}}$ //

For part (d) & (e) find the given sum :)

Recall: • $\sum_{i=1}^{n}i^2=\frac{n.(n+1)(2n+1)}{6}$

• $\sum_{i=1}^{n}r^{i-1}=\frac{r^n-1}{r-1}$

d) $\frac{1}{5}+\frac{4}{5}+\frac{9}{5}+\frac{16}{5}+\dots+45=?$ $\quad 45=\frac{1}{5}.(225)$, $225=15^2$

Soln: $\sum_{i=1}^{15}\frac{1}{5}i^2=\frac{1}{5}\sum_{i=1}^{15}i^2=\frac{1}{\cancel{5}}.\left(\frac{\cancel{15}.\cancel{16}.31}{\cancel{6}}\right)=248$ //

e) $1-\frac{1}{6}+\frac{1}{36}-\frac{1}{216}+\dots+\frac{1}{\underbrace{1679616}_{(-\frac{1}{6})^8}}=?$

Soln: $\sum_{i=1}^{9}\left(-\frac{1}{6}\right)^{i-1}=\frac{\left(-\frac{1}{6}\right)^9-1}{\left(-\frac{1}{6}\right)-1}$ //

Q1 : Evaluate the followings;

a) $\sum_{i=1}^{24}(2i-1)^2$

b) $\sum_{i=1}^{50}(\ln(i+3)-\ln(i+2))$

Recall; $\sum_{i=1}^{n} i = \frac{n.(n+1)}{2}$

c) $\sum_{i=1}^{20}\sin\left(\frac{i\pi}{2}\right)-\cos(i\pi)$

Solutions: (a) $\sum_{i=1}^{24}(2i-1)^2 = \sum_{i=1}^{24} 4i^2-4i+1 = 4\sum_{i=1}^{24} i^2 - 4\sum_{i=1}^{24} i + \sum_{i=1}^{24} 1$

$$= 4.\left(\frac{\cancel{24}^{\,4}.25.49}{\cancel{6}}\right) - 4.\left(\frac{\cancel{24}^{\,12}.25}{\cancel{2}}\right) + 24$$

$$= 18\,424$$

(b) $\underbrace{\sum_{i=1}^{50}(\ln(i+3)-\ln(i+2))}_{\ln 53 - \ln 3}$ $= \ln\left(\frac{53}{3}\right)$

For $i=1$: $\cancel{\ln 4} - \ln 3$

$i=2$: $\cancel{\ln 5} - \cancel{\ln 4}$

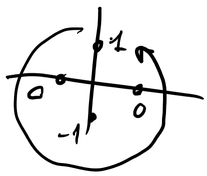
$i=3$: $\cancel{\ln 6} - \cancel{\ln 5}$

$\vdots$

$i=49$: $\ln 52 - \cancel{\ln 51}$

$+$ $i=50$: $\ln 53 - \cancel{\ln 52}$

$\ln 53 - \ln 3$



(c) $\sum_{i=1}^{20}\sin\left(\frac{i\pi}{2}\right)-\cos(i\pi)$

$$= \sin\left(\frac{\pi}{2}\right) - \underbrace{\cos(\pi)}_{-1} + \sin(\pi) - \cos(2\pi) + \sin\left(\frac{3\pi}{2}\right) - \cos(3\pi) + \sin(2\pi) - \cos(4\pi)$$
$$+ \dots + \sin(10\pi) - \cos(20\pi)$$

$$= \underbrace{\cancel{1}+\cancel{0}-\cancel{1}+\cancel{0}+\dots+0}_{0} + 1 - 1 + 1 - 1 + \dots - 1 = 0 + 0 = 0$$

Q3: Evaluate the followings

a) $\lim\limits_{n\to\infty} \sum\limits_{i=1}^{n} \frac{1}{(i+1)(i+2)}$

b) $\lim\limits_{n\to\infty} \sum\limits_{i=1}^{n} 4\left(1-\frac{i}{2n}\right)^2 \frac{1}{n^2}$

Soln:

$$\frac{1}{(i+1)(i+2)} = \frac{A}{\underset{(i+2)}{i+1}} + \frac{B}{\underset{(i+1)}{i+2}} = \frac{1}{i+1} - \frac{1}{i+2}$$

$i.0 + 1 = A(i+2) + B(i+1)$

$Ai + Bi = 0 \Rightarrow^{-1} A+B=0$

$2A+B = 1 \Rightarrow_{+} 2A+B=1$

$A = 1 \Rightarrow B=-1$

$(A+B)i = i.0$

$$\lim_{n\to\infty} \sum_{i=1}^{n} \frac{1}{i+1} - \frac{1}{i+2} = ?$$

$$= \lim_{n\to\infty}\left(\frac{1}{2} - \cancel{\frac{1}{3}} + \cancel{\frac{1}{3}} - \cancel{\frac{1}{4}} + \cancel{\frac{1}{4}} - \cancel{\frac{1}{5}} + \cdots + \cancel{\frac{1}{n}} - \cancel{\frac{1}{n+1}} + \cancel{\frac{1}{n+1}} - \frac{1}{n+2}\right)$$

$$= \lim_{n\to\infty}\left(\frac{1}{2} - \underbrace{\frac{1}{n+2}}_{\to 0}\right) = \frac{1}{2} //$$

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b) $\lim\limits_{n\to\infty} \sum\limits_{i=1}^{n} 4\left(1-\frac{i}{2n}\right)^2 \frac{1}{n^2}$

$$= \lim_{n\to\infty} \frac{4}{n^2} \sum_{i=1}^{n} 1 - \frac{\cancel{2}i}{\cancel{2}n} + \frac{i^2}{4n^2} = \lim_{n\to\infty} \frac{4}{n^2}\left(\sum_{i=1}^{n} 1 - \frac{i}{n} + \frac{i^2}{4n^2}\right)$$

$$= \lim_{n\to\infty} \frac{4}{n^2}\left(\sum_{i=1}^{n} 1 - \frac{1}{n}\sum_{i=1}^{n} i + \frac{1}{4n^2}\sum_{i=1}^{n} i^2\right)$$

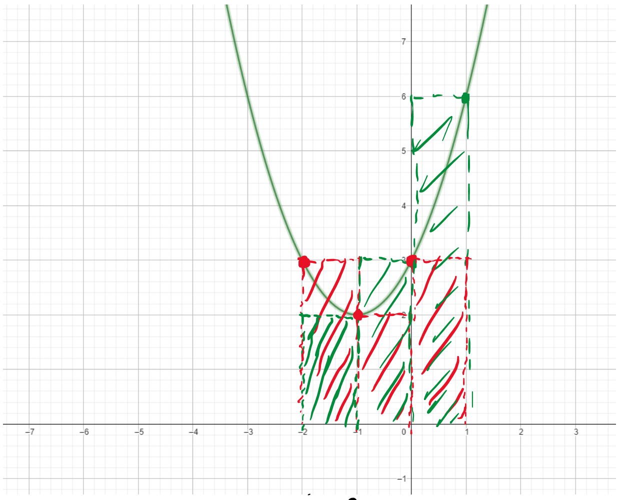
$$= \lim_{n\to\infty} \frac{4}{n^2}\left(n - \frac{1}{\cancel{n}}\cdot\frac{\cancel{n}.(n+1)}{2} + \frac{1}{4n^{\cancel{2}}}\cdot\frac{\cancel{n}.(n+1)(2n+1)}{6}\right)$$

$$= \lim_{n\to\infty} \frac{4}{n^2}\left(n - \frac{n+1}{2} + \frac{(n+1)(2n+1)}{24n}\right)$$

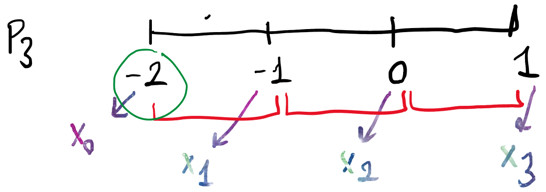
$$= \lim_{n\to\infty}\left(\underbrace{\frac{4}{n}}_{\to 0} - \underbrace{\frac{4.(n+1)}{2n^2}}_{\to 0} + \underbrace{\frac{4.(n+1)(2n+1)}{24n.n^2}}_{\to 0}\right)$$

$= 0 //$

Q4 : Find left and right endpoint sums corresponding to the partition P_n which is the division of an interval [a,b] into n-pieces of equal length



(a) $f(x) = x^2 + 2x + 3$, $n = 3$, $[-2,1]$



$$\Delta x = \frac{x_3 - x_0}{3} = \frac{1-(-2)}{3} = 1$$

Left Endpoint : (Red) $\sum_{i=1}^{3} f(x_{i-1}) \cdot \Delta x = f(x_0) \cdot 1 + f(x_1) \cdot 1 + f(x_2) \cdot 1$
$= f(-2) + f(-1) + f(0) = 3 + 2 + 3 = 8$

Right Endpoint : (Green) $\sum_{i=1}^{3} f(x_i) \cdot \Delta x = f(x_1) \cdot 1 + f(x_2) \cdot 1 + f(x_3) \cdot 1$
$= f(-1) + f(0) + f(1) = 2 + 3 + 6 = 11$

Q4(b)

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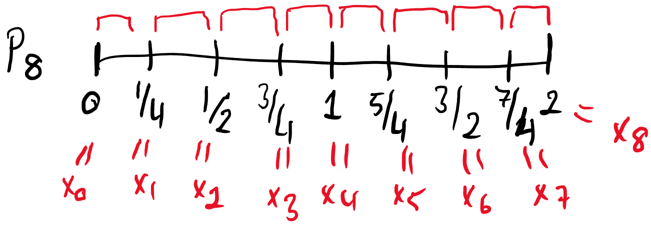
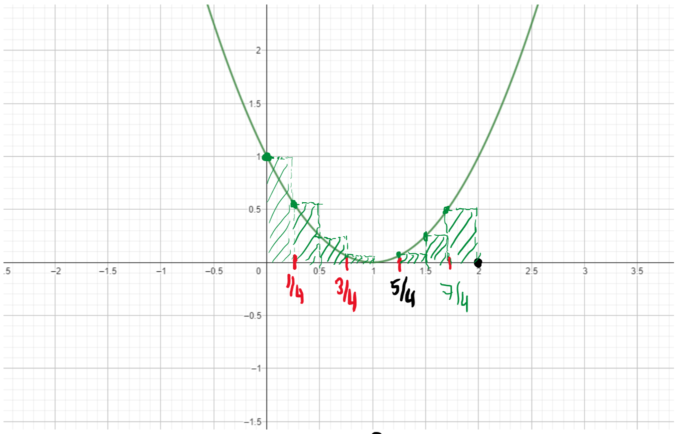
(b) $f(x) = x^2 - 2x + 1, \; n = 8, \; [0,2]$





$$\Delta x = \frac{x_8 - x_0}{8} = \frac{2-0}{8} = \frac{1}{4}$$

Left Endpoint: $\sum_{i=1}^{8} f(x_{i-1}) \cdot \Delta x = \Big(f(x_0) + f(x_1) + f(x_2) + f(x_3) + f(x_4) + f(x_5) + f(x_6) + f(x_7)\Big) \cdot \Delta x$

$$= \frac{1}{4}\left(f(0) + f\left(\tfrac{1}{4}\right) + f\left(\tfrac{1}{2}\right) + f\left(\tfrac{3}{4}\right) + f(1) + f\left(\tfrac{5}{4}\right) + f\left(\tfrac{3}{2}\right) + f\left(\tfrac{7}{4}\right)\right)$$

$f(x) = x^2 - 2x + 1 = (x-1)^2$

$$= \dots = \frac{1}{4}{}_{/4} + \frac{1}{8}{}_{/2} + \frac{5}{16} = \frac{11}{16}$$

Finding Right Endpoint sum is exercise :)

Q5

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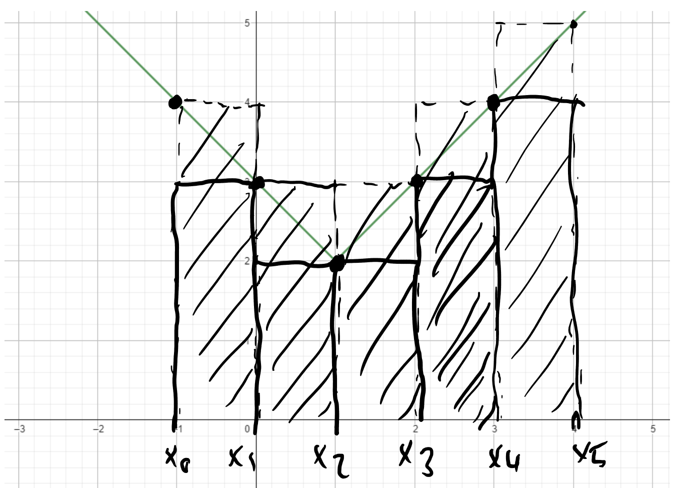
Q5 : Compute the upper and lower sum of the function $f(x) = 2 + |1-x|$ on $[-1,4]$, P_5. Compute the area under the graph of $f(x)$ and compare it with $\mathcal{U}(f,P_5)$, $\mathcal{L}(f,P_5)$. P_5 (equal length) !



Soln: $x > 1$, $f(x) = 2 + (-1+x) = 1 + x$

$x \leq 1$, $f(x) = 2 + 1 - x = 3 - x$

P_5: $\underset{x_0}{-1}$, $\;0 = x_1$, $\;1 = x_2$, $\;2 = x_3$, $\;3 = x_4$, $\;4 = x_5$

$$\Delta x = \Delta x_i = \frac{x_5 - x_0}{5} = \frac{4-(-1)}{5} = 1$$

$$\mathcal{U}(f,P_5) = \sum_{i=1}^{5} \sup_{t\in[x_i,x_{i+1}]} f(t)\,\Delta x_i = \big(f(x_0) + f(x_1) + f(x_3) + f(x_4) + f(x_5)\big)\cdot 1$$

$$= 4 + 3 + 3 + 4 + 5$$

$$= 19$$

$$\mathcal{L}(f,P_5) = \sum_{i=1}^{5} \inf_{t\in[x_i,x_{i+1}]} f(t)\,\Delta x_i = \big(f(x_1) + f(x_2) + f(x_2) + f(x_3) + f(x_4)\big)\cdot 1$$

$$= 3 + 2 + 2 + 3 + 4$$

$$= 14$$

$$\mathcal{L}(f,P_5) \leq \text{Area} \leq \mathcal{U}(f,P_5)$$

$$14 \leq \text{?Area} \leq 19$$

Q6

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