MATH - 120 - 2021 - 2 RECITATION PROBLEMS - WEEK 07

(1) Find the limits if they exist

1. $$\lim_{(x,y)\to(0,0)} \sqrt{x^2+y^2}\, \sin\left(\frac{1}{x^2+y^2}\right)$$

Soln:

We have

$$-\sqrt{x^2+y^2} \leq (x^2+y^2) \sin\left(\frac{1}{x^2+y^2}\right) \leq \sqrt{x^2+y^2}$$

for all $(x,y) \in \mathbb{R}^2 \setminus \{(0,0)\}$

$$\lim_{(x,y)\to(0,0)} -\sqrt{x^2+y^2} = 0 = \lim_{(x,y)\to(0,0)} \sqrt{x^2+y^2}$$

By Squeeze Theorem,

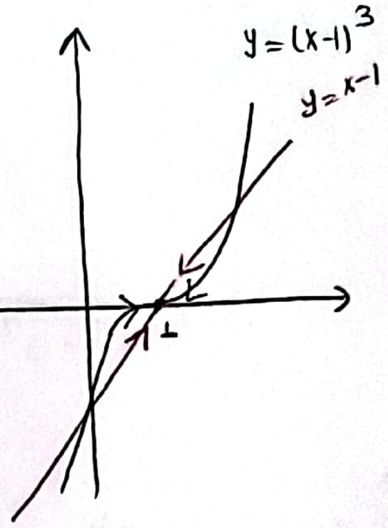
$$\lim_{(x,y)\to(0,0)} \sqrt{x^2+y^2}\, \sin\left(\frac{1}{x^2+y^2}\right) = 0$$

2. $\lim\limits_{(x,y)\to(1,0)} \dfrac{(x-1)^3 y}{((x-1)^6+y^2)}$

Soln:

Let $f(x,y) = \dfrac{(x-1)^3\, y}{(x-1)^6+y^2}$ be where

$\text{Dom} f = \mathbb{R}^2 - \{(1,0)\}$.



Approach $(0,0)$ along $y=(x-1)^3$

$$f(x, x^3+1) = \frac{(x-1)^3 \cdot (x-1)^3}{(x-1)^6 + (x-1)^6}$$

$$\lim_{\substack{(x,y)\to(1,0)\\ \text{along}\\ y=(x-1)^3}} f(x,y) = \lim_{x\to 1} \frac{(x-1)^6}{2\cdot(x-1)^6} = \frac{1}{2}$$

Approach $(0,0)$ along $y=x-1$

$$f(x,x-1) = \frac{(x-1)^3\cdot(x-1)}{(x-1)^6+(x-1)^2} = \frac{(x-1)^4}{(x-1)^6+(x-1)^2}$$

$$\lim_{\substack{(x,y)\to(1,0)\\ \text{along } y=x-1}} f(x,y) = \lim_{x\to 1} \frac{(x-1)^4}{(x-1)^6+(x-1)^2} = 0$$

Since $\frac{1}{2} \neq 0$, the limit does not exist

3. $\lim_{(x,y)\to(0,0)} \frac{x^2y^2}{x^2+y^4}$

Soln :

We have

$$0 < \frac{x^2y^2}{x^2+y^4} < \frac{x^2y^2}{x^2} = y^2$$

where $(x,y) \in \mathbb{R}^2 - \{(0,0)\}$

$$\lim_{(x,y)\to(0,0)} 0 = 0 = \lim_{(x,y)\to(0,0)} y^2$$

By Squeeze Theorem, $\lim_{(x,y)\to(0,0)} \frac{x^2y^2}{x^2+y^4} = 0.$

4. $$\lim_{(x,y)\to(0,1)} \frac{\sin\left(\pi\left(x^2+(y-1)^2\right)\right)}{|x|+|y-1|}$$

Soln:

Since $\pi\left(x^2+(y-1)^2\right) > 0$,

$$0 < \left| \frac{\sin\left(\pi\left(x^2+(y-1)^2\right)\right)}{|x|+|y-1|} \right| < \frac{\pi\left(|x|^2+|y-1|^2\right)}{|x|+|y-1|}$$

$$\lim_{(x,y)\to(0,0)} \frac{\pi\left(|x|^2+|y-1|^2\right)}{|x|+|y-1|} = ?$$

$$0 < \frac{\pi\left(|x|^2+|y-1|^2\right)}{|x|+|y-1|} = \frac{\pi\left(\left(|x|+|y-1|\right)^2 - 2|x||y-1|\right)}{|x|+|y-1|}$$

$$< \frac{\pi\cdot\left(|x|+|y-1|\right)^2}{|x|+|y-1|}$$

$$= \pi\cdot\left(|x|+|y-1|\right)$$

Since $\lim\limits_{(x,y)\to(0,0)} 0 = 0 = \lim\limits_{(x,y)\to(0,0)} \pi\cdot\left(|x|+|y-1|\right)$, by Squeeze Thm

$\lim\limits_{(x,y)\to(0,0)} \dfrac{\pi\left(|x|^2+|y-1|^2\right)}{|x|+|y-1|} = 0$. Thus, $\lim\limits_{(x,y)\to(0,0)} \dfrac{\sin\left(\pi\left(x^2+(y-1)^2\right)\right)}{|x|+|y-1|} = 0$

by Squeeze Thm.

(2) Determine when the following functions are continuous

1. $f(x,y) = \sin(x^2+y^2)\, e^{\frac{1}{\cos(x+y)+1}}$

Soln: The function f is continuous on

$\mathbb{R}^2 - \{(x,y) \in \mathbb{R}^2 \mid x+y = (2k+1)\pi,\ k \in \mathbb{Z}\}$ since it is

product of two continuous functions which are

trigonometric funct. and exponential function.

2. $$f(x,y) = \begin{cases} 0 & \text{if } (x,y) = (0,0) \\ \dfrac{\sin(|x|+|y|)}{2(|x|+|y|)} & \text{if } (x,y) \neq (0,0) \end{cases}$$

Soln:

The function f is continous on $\mathbb{R}^2 - \{(0,0)\}$

where $f(x,y) = \dfrac{\sin(|x|+|y|)}{2(|x|+|y|)}$

Now, let's check the continuouty of f at

$(x,y) = (0,0)$.

$$\lim_{(x,y)\to(0,0)} \frac{\sin(|x|+|y|)}{2(|x|+|y|)} = ?$$

Let $|x|+|y| = a$. If $(x,y)\to(0,0)$, then $a\to 0$

$$\lim_{(x,y)\to(0,0)} f(x,y) = \lim_{a\to 0} \frac{\sin(a)}{2a} = \frac{1}{2}$$

Since $\lim_{(x,y)\to(0,0)} f(x,y) = \frac{1}{2} \neq f(0,0)=0$, the function

f is not continuous at $(0,0)$.

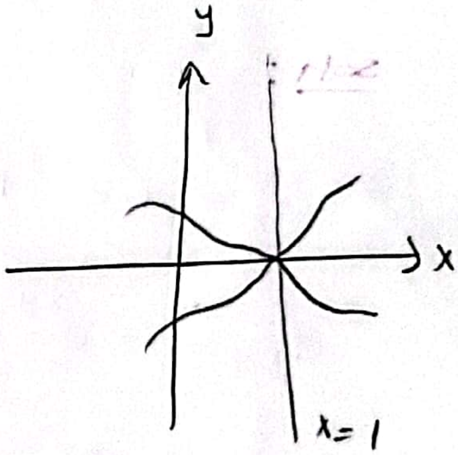
3.

$$f(x,y) = \begin{cases} x^2y & \text{if } x<1 \\ y^2(2x-1) & \text{if } x \geq 1 \end{cases}$$

soln:

The function f is continuous on the set $\{(x,y)\in\mathbb{R}^2 \mid x\in(-\infty,1)\cup(1,\infty)\}$ since f is the combination of the polynomials.

Now, let's check the continuity of f at the points $(1,a)$, $a\in\mathbb{R}$.



$$\lim_{\substack{(x,y)\to(1,a)\\(x<1)}} f(x,y) = \lim_{\substack{(x,y)\to(1,a)\\x<1}} x^2y = a$$

$$\lim_{\substack{(x,y)\to(1,a)\\(x>1)}} f(x,y) = \lim_{\substack{(x,y)\to(1,a)\\x>1}} y^2(2x-1) = a^2$$

So, $\lim\limits_{(x,y)\to(1,a)} f(x,y)$ exists if $a^2=a \iff a\cdot(a-1)=0$

$\iff a=0$ OR $a=1$.

So, f is not continuous at $(1,a)$ if $a\neq 0,1$.

Since $\lim\limits_{(x,y)\to(1,a)} f(x,y)$ does not exist if $a\neq 0,1$.

$\lim\limits_{(x,y)\to(1,0)} f(x,y) = 0 = f(1,0)$. and $\lim\limits_{(x,y)\to(1,1)} f(x,y) = 1 = f(1,1)$.

So, f is continuous at $(1,0)$ and $(1,1)$.

③ Evaluate

1. f_{xy} and f_{yx} where $f(x,y) = xe^{xy}$

<u>Soln</u>:

$$f_x = 1 \cdot e^{xy} + x \cdot e^{xy} \cdot y$$

$$f_{xy} = x \cdot e^{xy} + x^2 e^{xy} \cdot y + xe^{xy} = 2xe^{xy} + x^2 y e^{xy}$$

$$f_y = x^2 e^{xy}$$

$$f_{yx} = 2xe^{xy} + x^2 y e^{xy}$$

2. f_{xxy} where $f(x,y) = x^2 e^{\sin x} + x^2 \cos y$

Soln:

$$f_x = 2x e^{\sin x} + x^2 e^{\sin x} \cos x + 2x \cos y$$

$$= e^{\sin x} (2x + x^2 \cos x) + 2x \cos y$$

$$f_{xx} = e^{\sin x} \cos x (2x + x^2 \cos x) + e^{\sin x} (2 + 2x \cos x - x^2 \sin x)$$

$$+ 2 \cos y$$

$$f_{xxy} = -2 \sin y$$

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(4) Find $\dfrac{\partial f(0,0)}{\partial x}$ where $f(x,y) = \begin{cases} xy\sin\left(\frac{1}{x^2+y^2}\right) & \text{if } (x,y) \neq (0,0) \\ 0 & \text{if } (x,y) = (0,0) \end{cases}$

Is f_x continuous at $(x,y) = (0,0)$?

Soln :

$$f_x(0,0) = \lim_{h \to 0} \frac{f(0+h,0) - f(0,0)}{h}$$

$$= \lim_{h \to 0} \frac{h \cdot 0 \sin\frac{1}{h^2+0^2} - 0}{h} = \lim_{h \to 0} 0 = 0$$

Then,

$$f_x = \begin{cases} y\sin\left(\frac{1}{x^2+y^2}\right) + xy\cos\left(\frac{1}{x^2+y^2}\right) \cdot \left(\frac{-2x}{(x^2+y^2)^2}\right) & \text{if } (x,y) \neq (0,0) \\ 0 & \text{if } (x,y) = (0,0) \end{cases}$$

Let us check the continuity of f_x at $(x,y) = (0,0)$.

i.e $\displaystyle\lim_{(x,y) \to (0,0)} f_x(x,y) \stackrel{?}{=} 0$

$$\lim_{(x,y)\to(0,0)} f_x(x,y) = \lim_{\substack{(x,y)\to(0,0)\\ \text{on } x=0}} f_x(x,y) = \lim_{y\to 0} y\sin\left(\frac{1}{y^2}\right) = 0$$

$$\lim_{\substack{(x,y)\to(0,0)\\ \text{on } y=x}} f_x(x,y) = \lim_{x\to 0} x\sin\left(\frac{1}{2x^2}\right) - x^2\cos\left(\frac{1}{2x^2}\right)\cdot\frac{2x}{4x^4}$$

$$= \lim_{x\to 0} x\sin\left(\frac{1}{2x^2}\right) - \cos\left(\frac{1}{2x^2}\right)\cdot\frac{1}{2x}$$

$$= \text{d.n.e.}$$

$\Rightarrow \lim_{(x,y)\to(0,0)} f_x(x,y)$ d.n.e.

Hence, $f_x(x,y)$ is not continuous at $(0,0)$.

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