MATH-120 – RECITATION PROBLEMS – WEEK 06

(1) Find the equations of the planes satisfying the given conditions:

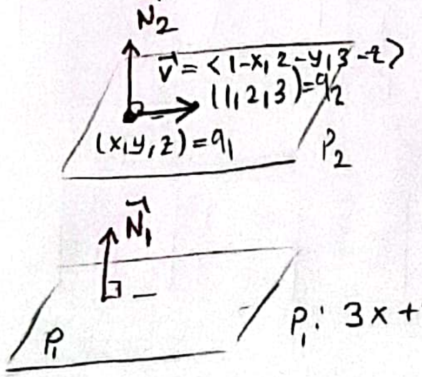
(a) Passing through $(1,2,3)$ and parallel to the plane $3x+y-2z=15$.

(b) Passing through the point $(-2,0,1)$ and the line of intersection of the planes $2x+3y-z=0$ and $x-4y+2z=-5$.

(c) Containing the line of intersection of ~~passing through the line~~ $x+y=2$, $y-z=3$ and perpendicular to the plane $2x+3y+4z=5$.

Soln:

(a)



$P_1: 3x+y-2z=15$

Since P_1 and P_2 are parallel, their normal vectors must be parallel.

The normal vector of P_1 is $\vec{N_1}=\langle 3,1,-2\rangle$ So, we can take $\vec{N_2}=\vec{N_1}=\langle 3,1,-2\rangle$.

Let $q_1 = (x, y, z)$ be a point on P_2. Then,

$\vec{v} = \vec{q_1 q_2} = \langle 1-x, 2-y, 3-z \rangle$ is perpendicular to $\vec{N_2}$.

So, $\vec{N_2} \cdot \vec{v} = 0 \iff \langle 3, 1, -2 \rangle \cdot \langle 1-x, 2-y, 3-z \rangle = 0$

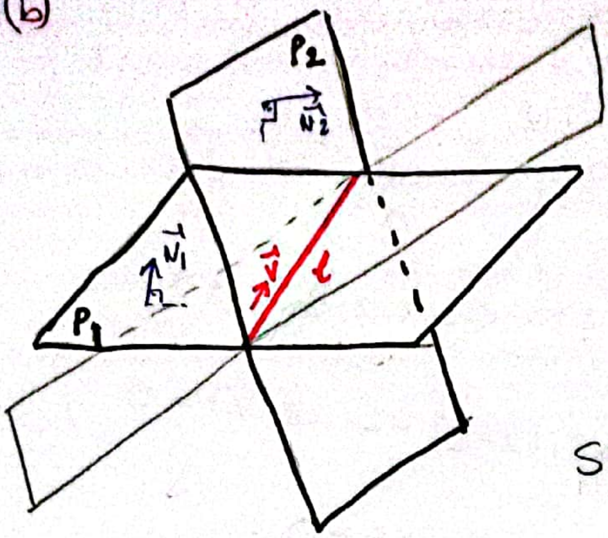
$\iff 3 \cdot (1-x) + 1 \cdot (2-y) - 2 \cdot (3-z) = 0$

$\iff 3 - 3x + 2 - y - 6 + 2z = 0$

$\iff \boxed{3x + y - 2z = -1}$

Thus, the plane equation of P_2 is $3x + y - 2z = -1$.

(b)



Let ℓ be the line of the intersection of the planes $P_1: 2x+3y-z=0$ and $P_2: x-4y+2z=-5$

$\vec{v}$: the direction vector of ℓ.

Since ℓ lies on P_1 and P_2,

$\vec{v} \perp \vec{n_1}$ and $\vec{v} \perp \vec{n_2}$ where $\vec{n_1}, \vec{n_2}$ are normal vectors of P_1 and P_2, respectively.

$\vec{n} = \langle 2, 3, -1 \rangle$ and $\vec{n_2} = \langle 1, -4, 2 \rangle$

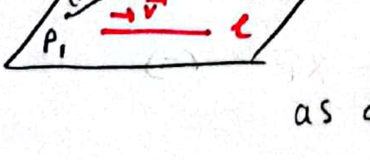
Since $\vec{v} \perp \vec{n_1}$ and $\vec{v} \perp \vec{n_2}$, $\vec{v} \parallel \vec{n_1} \times \vec{n_2}$.

Choose $\vec{v} = \vec{n_1} \times \vec{n_2} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ 2 & 3 & -1 \\ 1 & -4 & 2 \end{vmatrix} = \vec{i}\begin{vmatrix} 3 & -1 \\ -4 & 2 \end{vmatrix} - \vec{j}\begin{vmatrix} 2 & -1 \\ 1 & 2 \end{vmatrix} + \vec{k}\begin{vmatrix} 2 & 3 \\ 1 & -4 \end{vmatrix}$

$= \langle 2, -5, -11 \rangle$

$$\left.\begin{aligned} P_1 &: ① \ 2x+3y-z=0 \\ P_2 &: ② \ x-4y+2z=-5 \end{aligned}\right\} \quad \text{Choose } y=0. \quad \left.\begin{aligned} ① &\Rightarrow 2x-z=0 \\ ② &\Rightarrow x+2z=-5 \end{aligned}\right\} \Rightarrow \begin{aligned} x&=-1 \\ z&=-2 \end{aligned}$$

So, the point $A=(-1,0,-2) \in P_1 \cap P_2$, i.e. $A \in \ell$,



$\vec{AB} = \langle -1, 0, 1 \rangle$ and $\vec{v}$ on the desired plane P. So, we can take $\vec{n} = \vec{AB} \times \vec{v}$ as a normal vector of P.

$$\Rightarrow \vec{n} = \begin{vmatrix} i & j & k \\ -1 & 0 & 1 \\ 2 & -5 & -11 \end{vmatrix} = \vec{i}\begin{vmatrix} 0 & 1 \\ -5 & -11 \end{vmatrix} - \vec{j}\begin{vmatrix} -1 & 1 \\ 2 & -11 \end{vmatrix} + \vec{k}\begin{vmatrix} -1 & 0 \\ 2 & -5 \end{vmatrix}$$

$$= \langle 5, -9, 5 \rangle$$

Thus, the eqn of the plane P is

$$\vec{n} \cdot \langle x+2, y, z+1 \rangle = 0$$

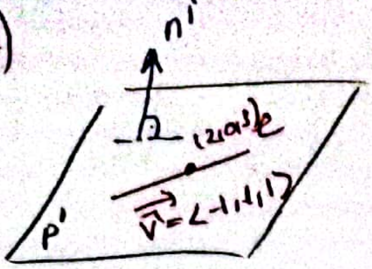
$$\Leftrightarrow \quad \langle 5, -9, 5 \rangle \cdot \langle x+2, y, z+1 \rangle = 0$$

$$P: \quad 5x + 10 - 9y + 5z + 5 = 0$$

$$\Leftrightarrow \quad 5x - 9y + 5z = 15.$$

CamScanner

c)



Let $l: \; x+y=2 \;,\; y-z=3$

Let $y=t$.

Then, $x=2-t$ and $z=t-3$

So, $l:\quad \begin{aligned} x&=2-t \\ y&=t \\ z&=-3+t \end{aligned} \quad$ for $t\in\mathbb{R}$.

$\Leftrightarrow \quad l: \langle 2,0,-3\rangle + t\cdot \underbrace{\langle -1,1,1\rangle}_{\vec{v}=\text{ direction vector of } l}$.

Say P' be a plane passing through the line l and perpendicular to the plane $P: 2x+3y+4z=5$.

$\vec{n}=\langle 2,3,4\rangle$ (normal vector of P)

Let $\vec{n'}=\langle a,b,c\rangle$ be a normal vector of P'.

Since $P\perp P'$, $\vec{n}\cdot\vec{n'}=0 \Leftrightarrow \langle 2,3,4\rangle\cdot\langle a,b,c\rangle=0$

$\Leftrightarrow \boxed{2a+3b+4c=0}$ ①

Since $\vec{v}\perp\vec{n'}$, $\vec{v}\cdot\vec{n'}=0 \Leftrightarrow \langle -1,1,1\rangle\cdot\langle a,b,c\rangle=0$

$\Leftrightarrow \boxed{-a+b+c=0}$ ②

$\Downarrow$

$b+c=a$

① $\Rightarrow 2\cdot(b+c)+3b+4c=0$

$\Rightarrow 5b=-6c \Rightarrow \left|\begin{aligned} b&=-6k \\ c&=5k \end{aligned}\right|, k\in\mathbb{R} \Rightarrow \boxed{a=-k}$

$\Rightarrow \vec{n'} = \langle a, b, c \rangle = \langle -k, -6k, 5k \rangle = k\langle -1, -6, 5 \rangle$

for some $k \in \mathbb{R}$.

Take $k=1 \Rightarrow \vec{n} = \langle -1, -6, 5 \rangle$ (normal vector of P').

Since $(2, 0, -3) \in \ell$ and $\ell \subset P'$, the plane equation

of P' is $\vec{n} \cdot \langle x-2, y, z+3 \rangle = 0$

$\Leftrightarrow \langle -1, -6, 5 \rangle \cdot \langle x-2, y, z+3 \rangle = 0$

$\Leftrightarrow -x + 2 - 6y + 5z + 15 = 0$

$\Leftrightarrow \boxed{-x - 6y + 5z = 17}$

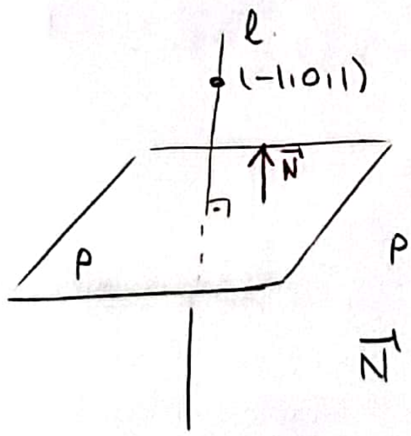
② Find equations of the line sepecified in vector and scalar parametric forms and in standard form.

(a) Through $(-1, 0, 1)$ and perpendicular to the plane $2x - y + 7z = 12$

(b) Through $(2, -1, -1)$ and parallel to the planes $x + y = 0$ and $x - y + 2z = 0$.

<u>Soln:</u>

a)



$P: 2x - y + 7z = 12$

$\vec{N} = \langle 2, -1, 7 \rangle$ is the normal vector of P.

$\vec{N}$ is parallel to the direction vector of the desired line. $v_1 \parallel \vec{N}$

position vector

$L: \langle x, y, z \rangle = \langle -1, 0, 1 \rangle + t \cdot \langle 2, -1, 7 \rangle$

$= \langle -1 + 2t, -t, 1 + 7t \rangle$ (vector eqn.)

$$\left.\begin{aligned} x &= -1 + 2t \\ y &= -t \\ z &= 1 + 7t \end{aligned}\right\} , t \in \mathbb{R} \quad \text{parametric eqn.}$$

$$t = \underbrace{\frac{x+1}{2} = \frac{y}{-1} = \frac{z-1}{7}}_{\text{standard form.}}$$

b)

If a line parallel to $x+y=0$ and $x-y+2z=0$, its direction vector is parallel to the cross product of the normal vectors to these two planes.

$\vec{N_1} = \langle 1,1,0 \rangle$ for $x+y=0$

$\vec{N_2} = \langle 1,-1,2 \rangle$ for $x-y+2z=0$

$$\vec{N_1} \times \vec{N_2} = \begin{vmatrix} i & j & k \\ 1 & 1 & 0 \\ 1 & -1 & 2 \end{vmatrix} = \vec{i}\begin{vmatrix} 1 & 0 \\ -1 & 2 \end{vmatrix} - \vec{j}\begin{vmatrix} 1 & 0 \\ 1 & 2 \end{vmatrix} + \vec{k}\begin{vmatrix} 1 & 1 \\ 1 & -1 \end{vmatrix}$$

$$= \vec{i}\cdot(2) - \vec{j}\,(2) + \vec{k}\,(-1-1)$$

$$= \langle 2,-2,-2 \rangle$$

So, we can take $\langle 2,-2,-2 \rangle$ as the direction vector of the desired line. Since the line passes through $(2,-1,-1)$, the desired line equation is

$$L: \langle x,y,z \rangle = \underbrace{\langle 2,-1,-1 \rangle}_{\text{position vector}} + t\cdot \langle 2,-2,-2 \rangle$$

$$= \langle 2+2t, -1-2t, -1-2t \rangle, \quad t \in \mathbb{R}$$

(in vector forms)

$$\left.\begin{aligned} x &= 2+2t \\ y &= -1-2t, \quad t\in\mathbb{R} \\ z &= -1-2t \end{aligned}\right\} \text{in scalar parametric form} \qquad t = \frac{x-2}{2} = \frac{y+1}{-2} = \frac{z+1}{-2} \Big\} \text{ in standard form}$$

③ Find the required distances:

(a) From $(1,2,0)$ to the plane $3x-4y-5z=2$

(b) From origin to the line $x+y+z=0,\ 2x-y-5z=1$

(c) Between the line $x-2=\frac{y+3}{2}=\frac{z-1}{4}$ and the plane $2y-z=1$ (First show that the line and the plane are parallel.)

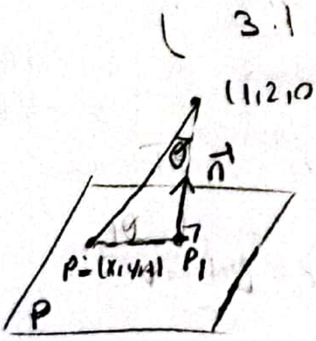
(d) Between the lines $L_1: x+2y=3\ ;\ y+2z=3$ and $L_2: x+y+z=6\ ,\ x-2z=-5$.

$$s=\frac{|Ax_0+By_0+Cz_0-D|}{\sqrt{A^2+B^2+C^2}}$$

Soln:

(a) The point $(1,2,0)$ does not lie on $3x-4y-5z=2$

$(3\cdot1-4\cdot2-5\cdot0=-6\neq2)$



$\overrightarrow{P_1P_0} \parallel \vec{n}=\langle 3,-4,-5\rangle$, $\overrightarrow{PP_0}=\langle 1-x,2-y,-z\rangle$

Let $s=|\overrightarrow{P_1P_0}|$ be. Then, s is the length of the projection of $\overrightarrow{PP_0}$ in the direction $\vec{n}$.

$P: 3x-4y-5z=2$

$$s=\left|\frac{\overrightarrow{PP_0}\cdot\vec{n}}{\|\vec{n}\|}\right|=\frac{|3-3x-8+4y+5z|}{\sqrt{3^2+4^2+5^2}}$$

$\overrightarrow{PP_0}\cdot\vec{n}=|\overrightarrow{PP_0}|\cdot|\vec{n}|\cos\theta$

$s=\left||\overrightarrow{PP_0}|\cos\theta\right|=\frac{|\overrightarrow{PP_0}\cdot\vec{n}|}{|\vec{n}|}$

$$=\frac{|-5-3x+4y+5z|}{\sqrt{50}}=\frac{|-7+\overbrace{2-3x+4y+5z}^{=0}|}{5\sqrt{2}}$$

$=7/5\sqrt{2}$

(b)

We have a line $x+y+z=0$ & $2x-y-5z=1$.

Then, $y = -x-z = 2x-5z-1$

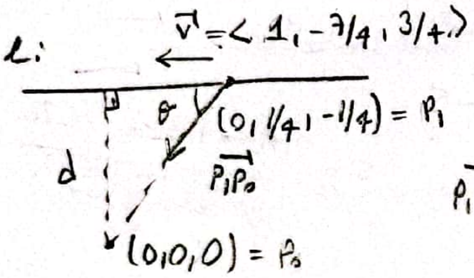
$\Rightarrow 3x-1 = 4z$.

Say $x=t$ for $t\in\mathbb{R}$. Then, $z = \frac{3t-1}{4}$ and $y = -t - \frac{3t-1}{4} = \frac{-7t+1}{4}$

Then, $\ell : \langle t, \frac{-7t+1}{4}, \frac{3t-1}{4} \rangle = t\langle 1, \frac{-7}{4}, \frac{3}{4} \rangle + \underbrace{\langle 0, \frac{1}{4}, \frac{-1}{4} \rangle}_{\vec{P_1}}$

$= t\cdot \underset{\uparrow}{\vec{v}} + \vec{P_1}$

direction vector



$\vec{P_1P_0} = \langle 0, -1/4, 1/4 \rangle$

$\|\vec{v}\times\vec{P_1P_0}\| = \|\vec{v}\|\cdot\|\vec{P_1P_0}\|\sin\theta = \|v\|\cdot\cancel{\|\vec{P_1P_0}\|}\cdot\frac{d}{\cancel{\|\vec{P_1P_0}\|}} \Rightarrow d = \frac{\|\vec{v}\times\vec{P_1P_0}\|}{\|v\|}$

$$\vec{v}\times\vec{P_1P_0} = \begin{vmatrix} i & j & k \\ 1 & -7/4 & 3/4 \\ 0 & -1/4 & 1/4 \end{vmatrix} = \vec{i}\begin{vmatrix} -7/4 & 3/4 \\ -1/4 & 1/4 \end{vmatrix} - \vec{j}\begin{vmatrix} 1 & 3/4 \\ 0 & 1/4 \end{vmatrix} + \vec{k}\begin{vmatrix} 1 & -7/4 \\ 0 & -1/4 \end{vmatrix}$$

$$= \vec{i}\left(-\frac{7}{16}+\frac{3}{16}\right) - \vec{j}\,(1/4) + \vec{k}\,(-1/4)$$

$$= \langle -1/4, -1/4, -1/4 \rangle$$

$\|\vec{v}\times\vec{P_1P_0}\| = \sqrt{1/16\cdot 3} = \frac{\sqrt{3}}{4}$ and $\|v\| = \sqrt{1+\frac{49}{16}+\frac{9}{16}} = \frac{\sqrt{74}}{4}$

So, $d = \frac{\sqrt{3}/4}{\sqrt{74}/4} = \frac{\sqrt{3}}{\sqrt{74}}$.

(c)

Firstly, let's find the direction vector of the given line:

$$L_1: \quad t = x-2 = \frac{y+3}{2} = \frac{z-1}{4}, \quad t \in \mathbb{R}.$$

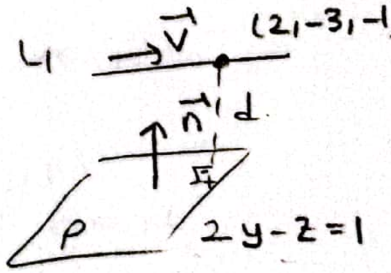
$$\Leftrightarrow \quad x = t+2, \quad y = 2t-3, \quad z = 4t-1$$

So, the direction vector of L_1 is

$$v = \langle 1, 2, 4 \rangle.$$

L_1 passing through the point $(2,-3,-1)$.

The plane $2y - z = 1$ has normal vector $\vec{n} = \langle 0, 2, -1 \rangle$.



$$\vec{v} \cdot \vec{n} = \langle 1, 2, 4 \rangle \cdot \langle 0, 2, -1 \rangle$$

$$= 0 + 4 - 4 = 0.$$

So, the line L_1 and the plane are parallel.

The distance from the line to the plane is equal to the distance from $(2,-3,1)$ to the plane $2y-z=1$

So is $\quad d = \dfrac{|2.(-3) - 1 - 1|}{\sqrt{2^2+1^2}} = \dfrac{8}{\sqrt{5}}.$

(d)

Fistly, consider the direction vectors of the given lines:

L_1

$L_1 : \begin{cases} x+2y=3 \\ y+2z=3 \end{cases}$ $\quad y = \frac{3-x}{2} = 3-2z = t \ , t\in\mathbb{R}.$

$$\left.\begin{aligned} x &= 3-2t \\ y &= t \\ z &= \frac{3-t}{2} \\ t&\in\mathbb{R} \end{aligned}\right\} \quad L_1 : \langle 3,0,3/2\rangle + t\langle -2,1,-1/2\rangle \qquad \vec{v_1} = \langle -2,1,-1/2\rangle$$

$L_2 : \begin{cases} x+y+z=6 \\ x-2z=-5 \end{cases}$ $\quad x = 6-y-z = -5+2z$

$\Rightarrow 3z = 6-y+5 = 11-y$

$z=t \Rightarrow y = 11-3t \ , t\in\mathbb{R}$

$x = 6-y-z = 6-(11-3t)-t = -5+2t$

So,
$$\left.\begin{aligned} x &= -5+2t \\ y &= 11-3t \\ z &= t \\ t&\in\mathbb{R} \end{aligned}\right\} \quad L_2 : (x,y,z) = (-5,11,0) + t(2,-3,1) \qquad \vec{v_2} = \langle 2,-3,1\rangle$$

L_1 and L_2 are not parallel since $\exists$ no λ s.t. $\vec{v_1} = \lambda.\vec{v_2}$.

$(L_1\cap L_2 = \emptyset)$

L_1 and L_2 does not intersect:

$(L_1\cap L_2 \neq \emptyset)$

On the other hand, assume that L_1 and L_2 intersect.

Let $(a,b,c)\in L_1\cap L_2$. Then, for some fixed $t,s\in\mathbb{R}$

$(a,b,c) = (3,0,3/2) + t(-2,1,-1/2) = (-5,11,0) + s.(2,-3,1)$

$$\left.\begin{aligned} &\Rightarrow 3-2t = -5+2s \\ &\Rightarrow t = 11-3s \end{aligned}\right\}$$

$2s+2t = 8 \rightarrow s+t=4$

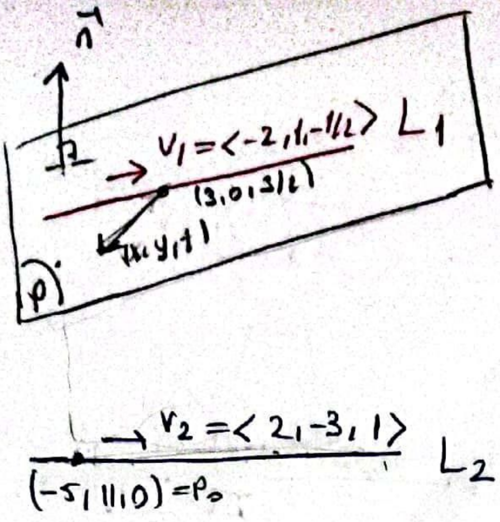
$t+3s = 11$

$2s = 7 \quad s = 7/2$

$t = 1/2$

$4-\frac{7}{2}$

$\Rightarrow 3/2 - t/2 = s \Rightarrow \frac{3}{2} - 1/4 \neq \frac{7}{2}$ $\lightning$



Now, let's consider the plane P which consists of the line L_1 and parallel to the line L_2.

So, $\vec{v_1} \perp \vec{n}$ and $\vec{v_2} \perp \vec{n}$. Then, $\vec{n} \parallel \vec{v_1} \times \vec{v_2}$

$$\vec{v_1} \times \vec{v_2} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ -2 & 1 & -1/2 \\ 2 & -3 & 1 \end{vmatrix} = \vec{i} \begin{vmatrix} 1 & -1/2 \\ -3 & 1 \end{vmatrix} - \vec{j} \begin{vmatrix} -2 & -1/2 \\ 2 & 1 \end{vmatrix} + \vec{k} \begin{vmatrix} -2 & 1 \\ 2 & -3 \end{vmatrix}$$

$$= \vec{i}\left(1 - \frac{3}{2}\right) - \vec{j}(-2+1) + \vec{k}(6-2)$$

$$= \langle -1/2, 1, 4 \rangle$$

Choose $\vec{n} = \vec{v_1} \times \vec{v_2} = \langle -1/2, 1, 4 \rangle$. Then, the plane equation P is $\langle -1/2, 1, 4 \rangle \cdot \langle x-3, y, z-\frac{3}{2} \rangle = 0$

$$\Leftrightarrow \; -\frac{x}{2} + \frac{3}{2} + y + 4z - 6 = 0$$

$$\Leftrightarrow \; -\frac{x}{2} + y + 4z = \frac{9}{2}$$

The distance between L_1 and L_2 is equal to the distance between any point on L_2 and the plane P since L_2 is parallel to P which contains L_1. Choose $(-5, 11, 0) \in L_2$.

$$S = \frac{\left| -\frac{1}{2}\cdot(-5) + 1\cdot 11 + 4\cdot 0 - \frac{9}{2} \right|}{\sqrt{\left(-\frac{1}{2}\right)^2 + 1^2 + 4^2}}$$

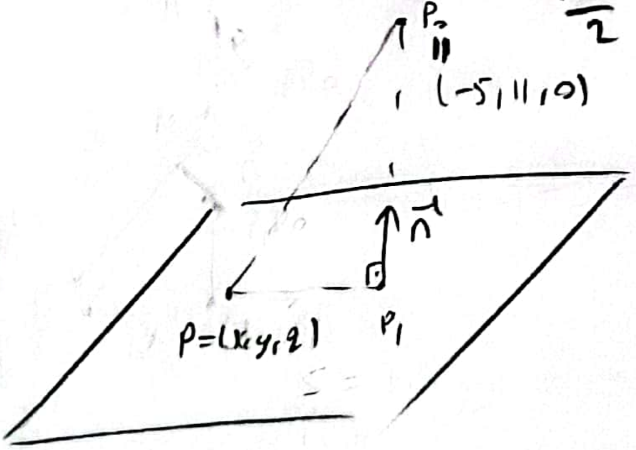
$$= \frac{9}{\sqrt{\frac{69}{4}}} = \frac{18}{\sqrt{69}}$$

$$S = \left| \frac{\vec{PP_0} \cdot \vec{n}}{|\vec{n}|} \right|$$

$$= \frac{\left| (-5-x)\cdot(-1/2) + (11-y)\cdot 1 + (-z)\cdot 4 \right|}{\sqrt{\left(-\frac{1}{2}\right)^2 + 1^2 + 4^2}}$$

$$= \frac{\left| \frac{5}{2} + \frac{x}{2} + 11 - y - 4z \right|}{\frac{\sqrt{69}}{2}} = \frac{\left| \frac{5}{2} - \frac{9}{2} + 11 \right| \cdot 2}{\sqrt{69}}$$

$$= \frac{18}{\sqrt{69}}$$



$$S = |\vec{P_1P_0}|$$