⑥ Consider the function defined by $f(x) = \int_0^x g(t)\,dt$

where

$$g(t) = \begin{cases} \dfrac{\sin(t)}{t} & \text{if } t \neq 0 \\ 1 & \text{if } t = 0. \end{cases}$$

Use the Maclaurin Series $\sin x = \sum_{n=0}^{\infty} (-1)^n \cdot \dfrac{x^{2n+1}}{(2n+1)!}$

to find the Maclaurin series of $f(x)$ and

$f^{(120)}(0)$ and $f^{(121)}(0)$.

<u>Soln:</u>

$$\sin t = \sum_{n=0}^{\infty} (-1)^n \frac{t^{2n+1}}{(2n+1)!} = t - \frac{t^3}{3!} + \frac{t^5}{5!} - \frac{t^7}{7!} + \cdots \quad , \text{ for } t \in \mathbb{R}$$

Then,

$$g(t) = \begin{cases} 1 - \dfrac{t^2}{3!} + \dfrac{t^4}{5!} - \dfrac{t^6}{7!} + \cdots & \text{if } t \neq 0 \\ 1 & \text{if } t = 0 \end{cases}$$

$$= 1 - \frac{t^2}{3!} + \frac{t^4}{5!} - \frac{t^6}{7!} + \cdots \quad \text{for } t \in \mathbb{R}.$$

$$= \sum_{n=0}^{\infty} (-1)^n \cdot \frac{t^{2n}}{(2n+1)!} \quad \forall t \in \mathbb{R}.$$

Then,

$$f(x) = \int_0^x g(t)\,dt = \int_0^x \sum_{n=0}^{\infty} \frac{(-1)^n \cdot t^{2n}}{(2n+1)!}\,dt$$

$$= \sum_{n=0}^{\infty} \frac{(-1)^n\, t^{2n+1}}{(2n+1)!\,(2n+1)} \Big|_0^x$$

$$= x - \frac{x^3}{3!\cdot 3} + \frac{x^5}{5!\,5} - \frac{x^7}{7!\cdot 7} + \dots \quad \text{for all } x \in \mathbb{R}$$

is the Maclaurin series of f and interval of convergence is $(-\infty, \infty)$. The radius of conv. is $R = \infty$

The Maclaurin series of f is

$$f(x) = \sum_{k=0}^{\infty} \frac{f^{(k)}(0)}{k!}(x-0)^k = \sum_{n=0}^{\infty} \frac{(-1)^n \cdot x^{2n+1}}{(2n+1)!\,(2n+1)}$$

$f^{(120)}(0) = 0$ Since the coeff of x^{2n}-terms are 0 $\forall n \geq 0$ in the Taylor series of f.

$f^{(121)}(0) = ?$ $k = 2n+1 = 121 \Rightarrow n = 60$

$$\frac{f^{(121)}(0)}{\cancel{121!}} = \frac{(-1)^{60}}{\cancel{121!}\,121} \Rightarrow f^{(121)}(0) = \frac{1}{121}.$$

(7) Use the Maclaurin series $e^x = \sum_{n=0}^{\infty} \frac{x^n}{n!}$ to approximate the integral

$$I = \int_0^1 x^2 e^{-(1/100)x^2} dx$$

with an error less than 10^{-3}.

Soln :

Since $\quad e^x = \sum_{n=0}^{\infty} \frac{x^n}{n!} \quad \forall x \in \mathbb{R},$

$$e^{-\frac{x^2}{100}} = \sum_{n=0}^{\infty} \frac{\left(-\frac{x^2}{100}\right)^n}{n!} = \sum_{n=0}^{\infty} (-1)^n \cdot \frac{1}{n!\,100^n} x^{2n}$$

multiply by "x^2":

$$x^2 e^{-\frac{1}{100}x^2} = \sum_{n=0}^{\infty} (-1)^n \cdot \frac{1}{n!\,100^n} x^{2n+2}$$

Integrating term by term we get:

$$\int x^2 e^{-1/100\, x^2} dx = \int \left(\sum_{n=0}^{\infty} (-1)^n \frac{1}{n!\,100^n} x^{2n+2} \right) dx$$

$$= \sum_{n=0}^{\infty} (-1)^n \frac{1}{n!\,100^n} \cdot \frac{x^{2n+3}}{2n+3} + C, \quad C \in \mathbb{R}$$

and therefore,

$$\int_0^1 x^2 e^{-\frac{1}{100}x^2} dx = \sum_{n=0}^{\infty} \frac{(-1)^n \cdot 1}{n!\, 100^n (2n+3)}$$

- It is an alternating since $a_{n+1} \cdot a_n < 0 \quad \forall n \geq 0$.

- $|a_{n+1}| = \dfrac{1}{(n+1)!\, 100^{n+1} (2n+5)} < \dfrac{1}{n!\, 100^n . (2n+3)} = |a_n| \quad \forall n \geq 0.$

So, $|a_n|$ is decreasing.

- $\sum_{n=0}^{\infty} |a_n| = \sum_{n=0}^{\infty} \dfrac{1}{n!\, 100^n (2n+3)}$ is convergent by Ratio test

$$\left(\lim_{n\to\infty} \frac{1}{\underset{n+1}{\cancel{(n+1)!}}\,.\,\underset{100}{\cancel{100^{n+1}}}(2n+5)} \cdot \frac{\cancel{n!} \cdot \cancel{100^n} . (2n+3)}{1} = \lim_{n\to\infty} \frac{2n+3}{100(n+1)(2n+5)} = 0 < 1 \right)$$

So, $\lim_{n\to\infty} |a_n| = 0$ by the n-th term test.

By AST, $\sum_{n=0}^{\infty} \dfrac{(-1)^n}{n!\, (2n+3) 100^n}$ is convergent

Now, we'll use alternating series error estimation

$$|Error| = |S - S_N| \le |a_{N+1}| = \frac{1}{(N+1)!\cdot 100^{N+1}(2N+5)} < 10^{-3} = \frac{1}{1000}$$

If $N=0$, $|Error| \le |a_1| = \frac{1}{1\cdot 100\cdot 5} = \frac{1}{500} > 10^{-3}$

If $N=1$, $|Error| \le |a_2| = \frac{1}{2!\cdot 100^2\cdot 7} < \frac{1}{1000}$

$$\frac{1}{2!\cdot 100^2\cdot 7} = \frac{1}{140000}$$

So, $|Error| < 10^{-3}$ for all $N \geq 1$.

Thus,

$$I = \int_0^1 x^2 e^{-0{,}01x^2}\,dx \approx a_0 + a_1 = \frac{1}{3} - \frac{1}{500}$$

Math 120 2021-2 RECITATION PROBLEMS - WEEK 05

① Find $\lim\limits_{x\to 0} \frac{1-e^{-x^2}}{x^2}$ without using L'Hopital

Soln:

Recall $e^x = \sum\limits_{n=0}^{\infty} \frac{x^n}{n!}$ for all $x \in \mathbb{R}$.

Then, $e^{-x^2} = \sum\limits_{n=0}^{\infty} (-1)^n \cdot \frac{x^{2n}}{n!} = 1 - \frac{x^2}{1!} + \frac{x^4}{2!} - \frac{x^6}{3!} + \dots$

$$\lim_{x\to 0} \frac{1-e^{-x^2}}{x^2} = \lim_{x\to 0} \frac{\cancel{1} - (\cancel{1} - x^2 + \frac{x^4}{2!} - \frac{x^6}{3!} + \dots)}{x^2}$$

$$= \lim_{x\to 0} \frac{x^2 + \frac{x^4}{2!} - \frac{x^6}{3!} + \dots}{x^2}$$

$$= \lim_{x\to 0} \frac{\cancel{x^2}(1 + \frac{x^2}{2!} - \frac{x^3}{3!} + \dots)}{\cancel{x^2}}$$

$$= \lim_{x\to 0} 1 + \frac{x^2}{2!} - \frac{x^3}{3!} + \dots$$

$$= 1.$$

(2) Describe (and sketch if possible) the set of points in $\mathbb{R}^3$ that satisfy the given equation or inequality.

(a) $x+y=1$

(b) $z \geq \sqrt{x^2+y^2}$

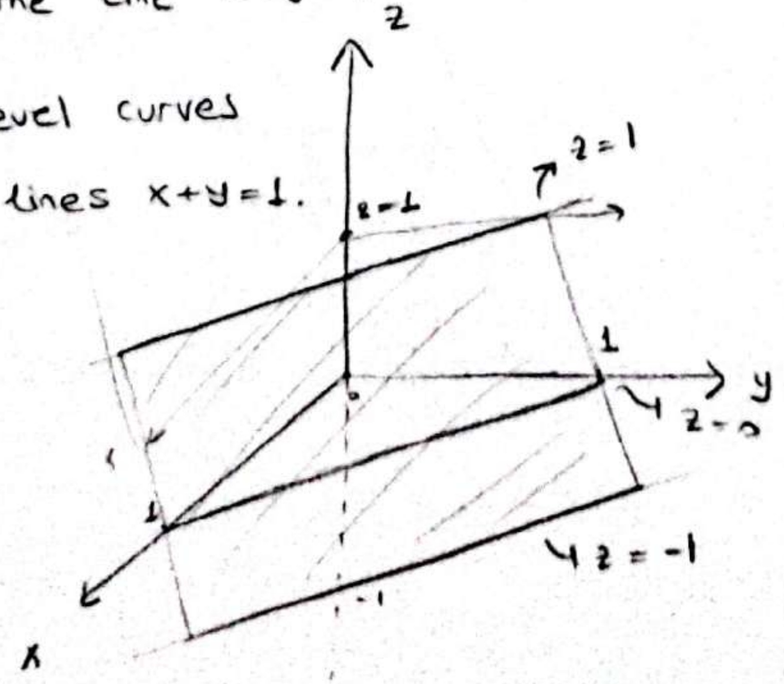
(c) $y \geq -1$ (optional)

Soln:

(a) $x+y=1$

$L = \{(x,y,z) \in \mathbb{R}^3 \mid x+y=1\}$ which is a plane in $\mathbb{R}^3$ s.t. there is the line $x+y=1$ at each $z=k \in \mathbb{R}$ level.

So, the level curves are the lines $x+y=1$.



b) $z \geq \sqrt{x^2+y^2}$

Firstly, consider $z = \sqrt{x^2+y^2}$.

$z = 0 \Rightarrow 0 = \sqrt{x^2+y^2} \Rightarrow x=0,\ y=0$

$z = 1 \Rightarrow 1 = \sqrt{x^2+y^2} \Rightarrow x^2+y^2 = 1$ (circle with $r=1$)

$z = 2 \Rightarrow 2 = \sqrt{x^2+y^2} \Rightarrow x^2+y^2 = 4$ (circle with $r=2$)

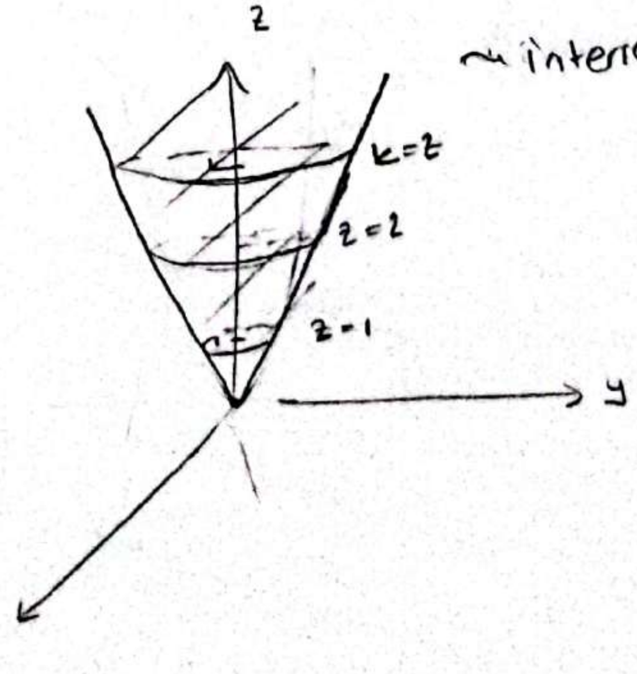
$\vdots$

$z = k \Rightarrow k = \sqrt{x^2+y^2} \Rightarrow x^2+y^2 = k^2$ (circle with $r=k$)

So, the level curves of surface at $z=k$ is the circle with the radius k.

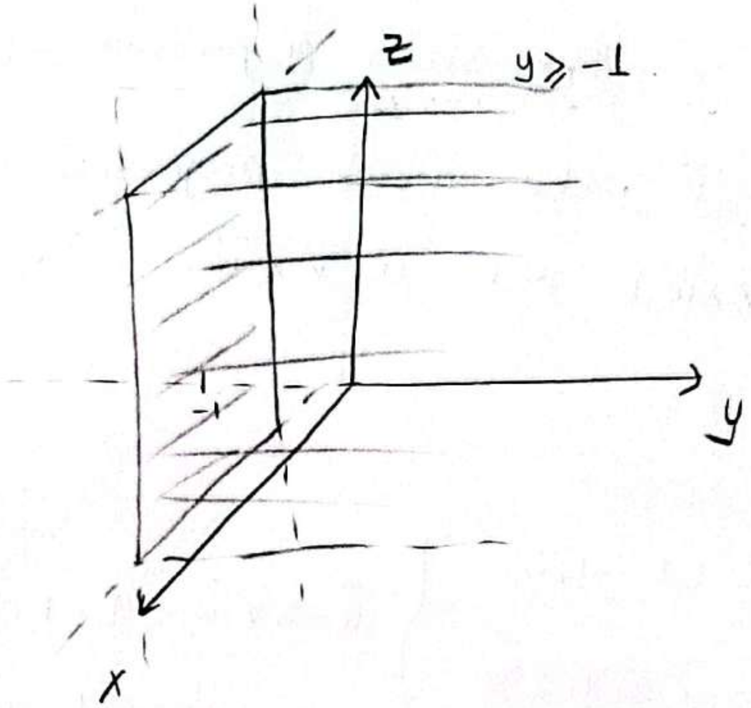
$z = \sqrt{x^2+y^2}$ is a cone in $\mathbb{R}^3$

$z \geq \sqrt{x^2+y^2}$ ~ interior of the cone.



(c) $y \geq -1$

Firstly consider $y=-1$ which is a plane in $\mathbb{R}^3$



③ Let $\vec{u} = \vec{i} - \vec{j}$, $\vec{v} = \vec{j} + 2\vec{k}$ and $\vec{w} = \vec{i} + 2\vec{j} + 3\vec{k}$

Find $\vec{u} - 3\vec{v}$, the length $\vec{v}$, unit vector in the direction of $\vec{v}$, the scolar projection of $\vec{u}$ in the direction of $\vec{v}$, the vector projection of $\vec{v}$ along $\vec{u}$, $\vec{u} \times (\vec{v} \times \vec{w})$ and $\vec{u} \cdot \vec{v} \times \vec{w}$.

Soln:

$$\left.\begin{aligned} \vec{u} &= \vec{i} - \vec{j} = \langle 1, -1, 0 \rangle \\ \vec{v} &= \vec{j} + 2\vec{k} = \langle 0, 1, 2 \rangle \end{aligned}\right\} \quad \begin{aligned} \vec{u} - 3\vec{v} &= \langle 1, -1, 0 \rangle - 3 \cdot \langle 0, 1, 2 \rangle \\ &= \langle 1, -1, 0 \rangle - \langle 0, 3, 6 \rangle \\ &= \langle 1, -4, -6 \rangle. \end{aligned}$$

$$\vec{w} = \vec{i} + 2\vec{j} + 3\vec{k} = \langle 1, 2, 3 \rangle$$

The length of $\vec{v} = |\vec{v}| = \| \langle 0, 1, 2 \rangle \| = \sqrt{0^2 + 1^2 + 2^2} = \sqrt{5}.$

Unit vector in the direction of $\vec{v} = \dfrac{\vec{v}}{|\vec{v}|} = \dfrac{\langle 0, 1, 2 \rangle}{\sqrt{5}}$

$$= \left\langle 0, \frac{1}{\sqrt{5}}, \frac{2}{\sqrt{5}} \right\rangle$$

The scolor projection of $\vec{u} = s = \dfrac{u \cdot v}{|v|}$

$$= \frac{\langle 1, -1, 0 \rangle \cdot \langle 0, 1, 2 \rangle}{\sqrt{5}}$$

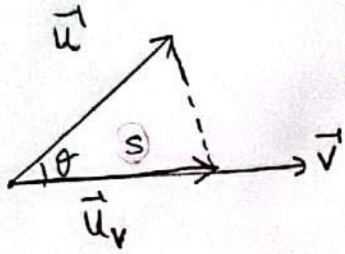
$$= \frac{0 - 1 + 0}{\sqrt{5}} = -\frac{1}{\sqrt{5}}$$

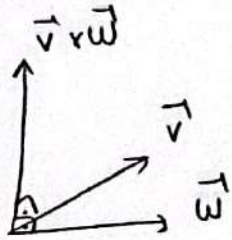
The vector projection of $\vec{v}$ along $\vec{u} = V_u = \dfrac{v \cdot u}{|u|^2} \cdot u$

$$u_v = \frac{\langle 0,1,2\rangle \cdot \langle 1,-1,0\rangle}{|\langle 1,-1,0\rangle|^2} \cdot \langle 1,-1,0\rangle$$

$$= \frac{-1}{2} \cdot \langle 1,-1,0\rangle$$

$$= \langle -\tfrac{1}{2}, \tfrac{1}{2}, 0\rangle$$





$\vec{u} \times (\vec{v} \times \vec{w}) = ?$

$$\vec{v} \times \vec{w} = \begin{vmatrix} i & j & k \\ 0 & 1 & 2 \\ 1 & 2 & 3 \end{vmatrix} = \vec{i} \begin{vmatrix} 1 & 2 \\ 2 & 3 \end{vmatrix} - \vec{j} \begin{vmatrix} 0 & 2 \\ 1 & 3 \end{vmatrix} + \vec{k} \begin{vmatrix} 0 & 1 \\ 1 & 2 \end{vmatrix}$$

$$= \vec{i}\,(3-4) - \vec{j}\,(0-2) + \vec{k}\,(0-1)$$

$$= -\vec{i} + 2\vec{j} - \vec{k}$$

$$= \langle -1, 2, -1\rangle$$

$$\vec{u} \times (\vec{v} \times \vec{w}) = \begin{vmatrix} i & j & k \\ 1 & -1 & 0 \\ -1 & 2 & -1 \end{vmatrix} = \vec{i} \begin{vmatrix} -1 & 0 \\ 2 & -1 \end{vmatrix} - \vec{j} \begin{vmatrix} 1 & 0 \\ -1 & -1 \end{vmatrix} + \vec{k} \begin{vmatrix} 1 & -1 \\ -1 & 2 \end{vmatrix}$$

$$= \hat{i}\,(1-0) - \hat{j}\,(-1-0) + \vec{k}\,(2-1)$$

$$= \hat{i} + \hat{j} + \vec{k}$$

$\vec{u} \cdot \vec{v} \times \vec{w} = ?$

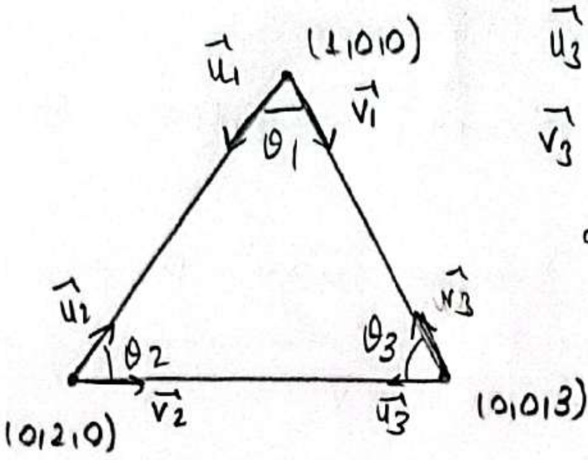
$\vec{v} \times \vec{w} = \langle -1, 2, -1 \rangle$

$\vec{u} \cdot (\vec{v} \times \vec{w}) = \langle 1, -1, 0 \rangle \cdot \langle -1, 2, -1 \rangle$

$= -1 - 2 + 0 = -3$

④ Find the three angles of the triangle with vertices $(1,0,0)$, $(0,2,0)$ and $(0,0,3)$.

<u>soln</u>:



$\vec{u_3} = -\vec{v_2} = \langle 0, 2, -3 \rangle$

$\vec{v_3} = -\vec{v_1} = \langle 1, 0, -3 \rangle$

$9 = \sqrt{13} \cdot \sqrt{10} \cos\theta_3$

$\Leftrightarrow \theta_3 = \arccos\left(\frac{9}{\sqrt{130}}\right)$

$\vec{u_1} = \langle 0-1, 2-0, 0-0 \rangle = \langle -1, 2, 0 \rangle$

$\vec{v_1} = \langle 0-1, 0-0, 3-0 \rangle = \langle -1, 0, 3 \rangle$

$\vec{u_1} \cdot \vec{v_1} = |u_1| \cdot |v_1| \cdot \cos\theta_1 \Leftrightarrow \langle -1, 2, 0 \rangle \cdot \langle -1, 0, 3 \rangle = |u_1| \cdot |v_1| \cos\theta_1$

$\Leftrightarrow 1 = \sqrt{(-1)^2 + 2^2 + 0^2} \cdot \sqrt{(-1)^2 + 0^2 + 3^2} \cdot \cos\theta_1$

$\Leftrightarrow 1 = \sqrt{5} \cdot \sqrt{10} \cos\theta_1$

$\Leftrightarrow \cos\theta_1 = \frac{1}{5\sqrt{2}}$

$\Leftrightarrow \theta_1 = \arccos\left(\frac{1}{5\sqrt{2}}\right)$.

$\vec{u_2} = \langle 1-0, 0-2, 0-0 \rangle = \langle 1, -2, 0 \rangle$

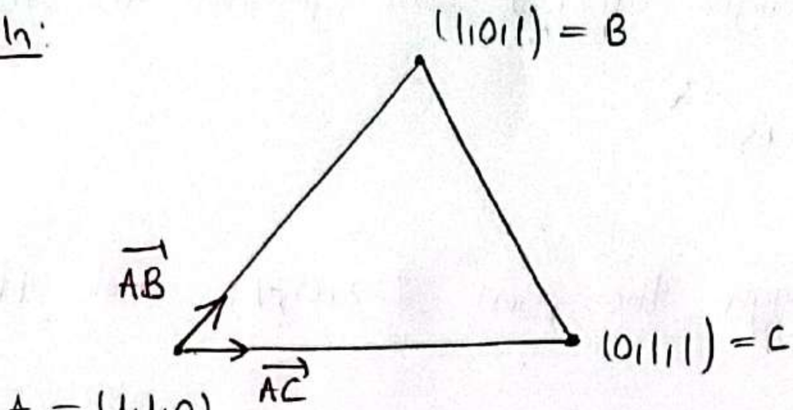
$\vec{v_2} = \langle 0-0, 0-2, 3-0 \rangle = \langle 0, -2, 3 \rangle$

$4 = \sqrt{5} \cdot \sqrt{13} \cos\theta_2 \Rightarrow \theta_2 = \arccos\left(\frac{4}{\sqrt{65}}\right)$

(5) Find the area of the triangle with vertices

$(1,1,0)$, $(1,0,1)$ and $(0,1,1)$.

Soln:



$$\vec{AB} = \langle 1-1, 0-1, 1-0 \rangle = \langle 0, -1, 1 \rangle$$

$$\vec{AC} = \langle 0-1, 1-1, 1-0 \rangle = \langle -1, 0, 1 \rangle$$

The area of the triangle is

$$\frac{1}{2} | \vec{AB} \times \vec{AC} | = \frac{1}{2} \left| \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ 0 & -1 & 1 \\ -1 & 0 & 1 \end{vmatrix} \right|$$

$$= \frac{1}{2} \left| \vec{i} \cdot \begin{vmatrix} -1 & 1 \\ 0 & 1 \end{vmatrix} - \vec{j} \begin{vmatrix} 0 & 1 \\ -1 & 1 \end{vmatrix} + \vec{k} \begin{vmatrix} 0 & -1 \\ -1 & 1 \end{vmatrix} \right|$$

$$= \frac{1}{2} \left| \vec{i}(-1) - \vec{j}(0-(-1)) + \vec{k}(0-1) \right|$$

$$= \frac{1}{2} |-\vec{i} - \vec{j} - \vec{k}| = \frac{1}{2} \sqrt{(-1)^2 + (-1)^2 + (-1)^2}$$

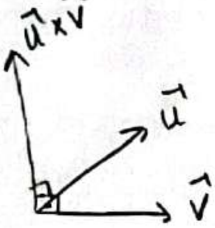
$$= \frac{1}{2} \cdot \sqrt{3}$$

(4) Find a unit vector perpendicular to the vectors

$\vec{i}+\vec{j}$ and $\vec{j}+2\vec{k}$.

<u>Soln</u>

Let say $\vec{u}=\vec{i}+\vec{j}=\langle 1,1,0\rangle$ and $\vec{v}=\vec{j}+2\vec{k}=\langle 0,1,2\rangle$



The vector $\vec{u}\times\vec{v}$ is perpendicular to both $\vec{u}$ and $\vec{v}$.

$$\vec{u}\times\vec{v}=\begin{vmatrix} i & j & k \\ 1 & 1 & 0 \\ 0 & 1 & 2 \end{vmatrix}=\vec{i}\begin{vmatrix} 1 & 0 \\ 1 & 2 \end{vmatrix}-\vec{j}\begin{vmatrix} 1 & 0 \\ 0 & 2 \end{vmatrix}+\vec{k}\begin{vmatrix} 1 & 1 \\ 0 & 1 \end{vmatrix}$$

$$=\vec{i}(2-0)-\vec{j}(2-0)+\vec{k}(1-0)$$

$$=2\vec{i}-2\vec{j}+\vec{k}$$

$$=\langle 2,-2,1\rangle$$

$$|\vec{u}\times\vec{v}|=\sqrt{2^2+(-2)^2+1^2}=\sqrt{9}=3\neq 1$$ (not unit)

The unit vector in the direction of $\vec{u}\times\vec{v}$ is

$$\frac{\vec{u}\times\vec{v}}{|\vec{u}\times\vec{v}|}=\frac{\langle 2,-2,1\rangle}{3}=\left\langle \frac{2}{3},-\frac{2}{3},\frac{1}{3}\right\rangle$$