① Determine the power series representations of the following functions. On what intervals is each representation valid (converges to the value of the function)?

(a) $\frac{1}{120-x}$ in powers of x

Soln: $\boxed{\frac{1}{1-x} = \sum_{n=0}^{\infty} x^n \quad \text{if } |x|<1}$

$$f(x) = \frac{1}{120-x} = \frac{1}{120\left(1-\frac{x}{120}\right)} = \frac{1}{120}\,\frac{1}{1-\frac{x}{120}} = \frac{1}{120}\sum_{n=0}^{\infty}\left(\frac{x}{120}\right)^n \quad \text{if } \left|\frac{x}{120}\right|<1$$

$$= \frac{1}{120}\sum \frac{x^n}{120^n}$$

$$= \sum_{n=0}^{\infty} \frac{x^n}{120^{n+1}} \quad \text{if } |x|<120$$

Let's check the convergence of $\sum_{n=0}^{\infty} \frac{x^n}{120^{n+1}}$ when $x=-120$ and $x=120$:

- If $x=-120$, then we'll consider $\sum_{n=0}^{\infty} \frac{(-120)^n}{120^{n+1}} = \sum_{n=0}^{\infty} \frac{(-1)^n}{120}$

Say $b_n = \frac{(-1)^n}{120}$. $\lim_{n\to\infty} b_{2n} = \frac{1}{120} \neq \lim_{n\to\infty} b_{2n+1} = -\frac{1}{120}$

$\Rightarrow \lim_{n\to\infty} b_n$ d.n.e. By the "nth term test", $\sum_{n=0}^{\infty} \frac{(-1)^n}{120}$ diverges

• If $x = 120$, then we have $\sum_{n=0}^{\infty} \frac{(120)^n}{120^{n+1}} = \sum_{n=0}^{\infty} \frac{1}{120}$

which is also divergent by the "nth term test"

(Say $c_n = \frac{1}{120}$. $\lim_{n \to \infty} \frac{1}{120} = \frac{1}{120} \neq 0$.)

Thus, $\sum_{n=0}^{\infty} \frac{x^n}{120^{n+1}}$ converges to $f(x) = \frac{1}{120-x}$ if

$-120 < x < 120$.

b) $\frac{1}{(120-x)^2}$ in powers of x.

Soln:

$$\frac{1}{120-x} = \sum_{n=0}^{\infty} \frac{x^n}{120^{n+1}} \quad \text{for } |x|<120$$

Take the derivative both sides:

$$\frac{1}{(120-x)^2} = \sum_{n=1}^{\infty} \frac{n \cdot x^{n-1}}{120^{n+1}}, \quad |x|<120$$

End points:

- If $x=120$, then $\sum_{n=1}^{\infty} \frac{n \cdot (120)^{n-1}}{120^{n+1}} = \sum_{n=1}^{\infty} n \cdot \frac{1}{120^2}$ is divergent by the n-th term test.
- If $x=-120$, then $\sum_{n=1}^{\infty} \frac{n \cdot (-120)^{n-1}}{120^{n+1}} = \sum_{n=1}^{\infty} (-1)^{n-1} \cdot \frac{n}{120^2}$ is divergent by the n-th term test.

$$\therefore \sum_{n=1}^{\infty} n \cdot \frac{x^{n-1}}{120^{n+1}} = \frac{1}{(120-x)^2} \quad \text{if } |x|<120.$$

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(c) $\ln(x)$ in powers of $x-120$

<u>Soln:</u>

$$\frac{1}{1-x} = \sum_{n=0}^{\infty} x^n \quad \text{if } |x|<1$$

$$\frac{1}{x} = \frac{1}{x-120+120} = \frac{1}{120\left(1+\frac{x-120}{120}\right)} = \frac{1}{120}\sum_{n=0}^{\infty}\left(-\frac{(x-120)}{120}\right)^n$$

$$1-\left(1-\frac{(x-120)}{120}\right)$$

$$= \sum_{n=0}^{\infty} (-1)^n \cdot \frac{(x-120)^n}{120^{n+1}} \qquad 1-\left(-\frac{(x+120)}{120}\right)$$

$$\Rightarrow \frac{1}{x} = \sum_{n=0}^{\infty} (-1)^n \frac{(x-120)^n}{120^{n+1}} \quad \text{if } \left|\frac{x-120}{120}\right| < 1$$

Take integral both sides:

$$\cdot \ln|x| = \sum_{n=0}^{\infty} (-1)^n \cdot \frac{1}{120^{n+1}} \frac{(x-120)^{n+1}}{n+1} + C \quad \text{if } |x-120| < 120$$

$$\Updownarrow$$

$$\boxed{0 < x < 240}$$

$x=120 \Rightarrow C = \ln 120$

• If $x=0$, then $\ln 120 + \sum_{n=0}^{\infty} (-1)^n \frac{1}{120^{n+1}} \cdot \frac{(-120)^{n+1}}{n+1} = \sum_{n=0}^{\infty} \underbrace{(-1)^{2n+1}}_{-1} \frac{1}{n+1} + \ln 120$

which is divergent (harmonic series)

• If $x=240$, then $\ln 120 + \sum_{n=0}^{\infty} (-1)^n \cdot \frac{1}{120^{n+1}} \cdot \frac{(120)^{n+1}}{n+1} = \sum_{n=0}^{\infty} (-1)^n \cdot \frac{1}{n+1} + \ln 120$

which is convergent by AST.

$\therefore \ln|x| = \sum_{n=0}^{\infty} (-1)^n \cdot \frac{(x-120)^{n}}{120^{n+1}} + \ln 120$ if. $0 < x \leq 240$.

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① For each of the following series, find the sum.

a) $\sum_{n=3}^{\infty} \frac{1}{n \cdot 2^n}$

Soln:

$$\ln|1-x| = \sum_{n=1}^{\infty} -\frac{x^n}{n} \quad \text{for } |x|<1 \quad (*)$$

$$\sum_{n=3}^{\infty} \frac{1}{n \cdot 2^n} = \left(\sum_{n=1}^{\infty} \frac{1}{n \cdot 2^n}\right) - \left(\frac{1}{2} + \frac{1}{8}\right)$$

$$= -\frac{5}{8} + \sum_{n=1}^{\infty} \frac{1}{n \cdot 2^n}$$

If $x = 1/2 < 1$ then $\sum_{n=1}^{\infty} \frac{1}{n \cdot 2^n} = -\ln\left|1-\frac{1}{2}\right| = -\ln|1/2|$

So, $\sum_{n=3}^{\infty} \frac{1}{n \cdot 2^n} = -\frac{5}{8} - \ln(1/2)$.

(*) $$\frac{1}{1-x} = \sum_{n=0}^{\infty} x^n \quad \text{if } |x|<1$$

Take integral both sides:

$$-\ln|1-x| = \sum_{n=0}^{\infty} \frac{x^{n+1}}{n+1} + C \Rightarrow \boxed{\ln|1-x| = \sum_{n=1}^{\infty} -\frac{x^n}{n}}$$

If $x=0 \Rightarrow C=0$

b) $\sum_{n=2}^{\infty} \frac{n^2}{2^n}$

Soln:

$$\begin{cases} 1+x+x^2+x^3+\cdots \\ 0+1+2x+3x^2+\cdots \end{cases}$$

$$\frac{1}{1-x} = \sum_{n=0}^{\infty} x^n \quad \text{for } |x|<1$$

Take the derivative of both sides:

$$x \cdot \frac{1}{(1-x)^2} = x \cdot \sum_{n=1}^{\infty} n \cdot x^{n-1} \quad \text{for } |x|<1$$

$$\Rightarrow \frac{x}{(1-x)^2} = \sum_{n=1}^{\infty} n \cdot x^n \quad \text{for } |x|<1$$

Take the derivative of both sides:

$$x \cdot \frac{1 \cdot (1-x)^{\cancel{2}\,1} - x \cdot 2 \cdot \cancel{(1-x)} \cdot (-1)}{(1-x)^{\cancel{4}\,3}} = x \cdot \sum_{n=1}^{\infty} n^2 \cdot x^{n-1} \quad \text{for } |x|<1$$

$$\Rightarrow x \cdot \frac{(1-x)+2x}{(1-x)^3} = \sum_{n=1}^{\infty} n^2 x^n \quad \text{for } |x|<1$$

If $x=\frac{1}{2}<1$, then $\underbrace{\frac{1}{2} \cdot \frac{(1-\frac{1}{2})+2\cdot\frac{1}{2}}{(1-\frac{1}{2})^3}}_{=6} = \sum_{n=1}^{\infty} n^2 \left(\frac{1}{2}\right)^n$

$$\Rightarrow \underbrace{\sum_{n=1}^{\infty} n^2 \left(\frac{1}{2}\right)^n}_{= 6} = \frac{1}{2} + \sum_{n=2}^{\infty} n^2 \left(\frac{1}{2}\right)^n \Rightarrow \sum_{n=2}^{\infty} n^2 \left(\frac{1}{2}\right)^n = 6 - \frac{1}{2} = \frac{11}{2}.$$

(3) Determine the first three terms of ~~the~~ Maclaurin series of $f(x)=(1+x)^{\alpha}$

where $\alpha \in \mathbb{R}$.

Soln:

Recall (Taylor and Maclaurin Series)

If $f(x)$ has derivative of all orders at $x=c$

(i.e., if $f^{(k)}(c)$ exists for $k=0,1,2,\ldots$), then the

series $$\sum_{k=0}^{\infty} \frac{f^{(k)}(c)}{k!}(x-c)^k = f(c)+f'(c)(x-c) + \frac{f''(c)}{2!}(x-c)^2 + \frac{f^{(3)}(c)}{3!}(x-c)^3 + \ldots$$

is called the Taylor series of f about c.

If $c=0$, the term Maclaurin series is usually used

in place of Taylor series.

Consider $f(x) = (1+x)^{\alpha}$, $\alpha, \beta \in \mathbb{R}$ that has a derivative of all orders at $x=0$.

$$f(x) = (1+x)^{\alpha} \Rightarrow f(0) = 1$$

$$f'(x) = \alpha \cdot (1+x)^{\alpha-1} \Rightarrow f'(0) = \alpha$$

$$f''(x) = \alpha \cdot (\alpha-1)\,(1+x)^{\alpha-2} \Rightarrow f''(0) = \alpha \cdot (\alpha-1)$$

$$f'''(x) = \alpha \cdot (\alpha-1)(\alpha-2)(1+x)^{\alpha-3} \Rightarrow f'''(0) = \alpha \cdot (\alpha-1)(\alpha-2)$$

$$\vdots$$

$$f^{(k)}(x) = \alpha \cdot (\alpha-1) \cdots (\alpha-(k-1))\,(1+x)^{\alpha-k} \Rightarrow f^{(k)}(0) = \alpha \cdot (\alpha-1) \cdots (\alpha-k+1)$$

Then, the Maclaurin series of f is

$$\sum_{k=0}^{\infty} \frac{\alpha \cdot (\alpha-1) \cdots (\alpha-k+1)}{k!} (x-0)^k$$

$$= \underbrace{1 + \frac{\alpha}{1!} x + \frac{\alpha \cdot (\alpha-1)}{2!} x^2}_{\text{the first three terms}} + \frac{\alpha \cdot (\alpha-1)(\alpha-2)}{3!} x^3 + \cdots$$

for $\alpha \in \mathbb{R}$

④ Storting from the formula

$$\frac{1}{1-x} = \sum_{n=0}^{\infty} x^n \quad \text{for } |x|<1,$$

find the Maclaurin series of the following functions, and determine their intervals of convergence:

(a) $f(x) = \dfrac{3}{x^2+4}$

soln:

$$f(x) = \frac{3}{x^2+4} = \frac{3}{4\left(\frac{x^2}{4}+1\right)} = \frac{3}{4} \cdot \frac{1}{1-\left(-\frac{x^2}{4}\right)}$$

Then,

$$f(x) = \frac{3}{4} \cdot \frac{1}{1-\left(-\frac{x^2}{4}\right)} = \frac{3}{4} \cdot \sum_{n=0}^{\infty} \left(-\frac{x^2}{4}\right)^n = \frac{3}{4} \sum_{n=0}^{\infty} \frac{(-1)^n}{4^n} \cdot x^{2n}$$

$$= \frac{3}{4}\left[1 - \frac{1}{4}x^2 + \frac{1}{4^2}x^4 - \frac{1}{4^3}x^6 + \frac{1}{4^4}x^8 - \dots\right]$$

if $\left|-\frac{x^2}{4}\right| < 1 \iff |x^2| < 4 \iff 0 < x^2 < 4$

$\iff -2 < x < 2$

So, the Maclaurin series of f is $\sum_{n=0}^{\infty} \frac{(-1)^n \cdot 3}{4^{n+1}} x^{2n}$

if $-2 < x < 2$.

- If $x = -2$, then $\sum_{n=0}^{\infty} \frac{(-1)^n \cdot 3}{4^{n+1}} (-2)^{2n} = \sum_{n=0}^{\infty} \frac{(-1)^n \cdot 3}{4^{n+1}} \cdot 4^n = \sum_{n=0}^{\infty} (-1)^n \frac{3}{4}$

is divergent by the "nth term test".

- If $x = 2$, then $\sum_{n=0}^{\infty} \frac{(-1)^n \cdot 3}{4^{n+1}} \cdot (2)^{2n} = \sum_{n=0}^{\infty} (-1)^n \cdot \frac{3}{4}$ is divergent

by the "n-th term test".

So, the interval of convergence is $(-2, 2)$.

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(b) $f(x) = x \ln(1+2x)$

soln :

$$\frac{1}{1-x} = \sum_{n=0}^{\infty} x^n \quad \text{if } |x|<1$$

Take the integral both sides:

$$-\ln|1-x| = \sum_{n=0}^{\infty} \frac{x^{n+1}}{n+1} = \sum_{n=1}^{\infty} \frac{x^n}{n} \quad \text{if } |x|<1$$

$$\Rightarrow \ln|1-x| = \sum_{n=1}^{\infty} -\frac{x^n}{n} \quad \text{if } |x|<1$$

Then,

$$\underset{\substack{\| \\ 1-(-2x)}}{\ln|1+2x|} = \sum_{n=1}^{\infty} -\frac{(-2x)^n}{n} \quad \text{if } |-2x|<1$$

$$= \sum_{n=1}^{\infty} (-1)^{n+1} \frac{2^n \cdot x^n}{n}$$

Multiply by "x" both sides:

$$f(x) = x \cdot \ln|1+2x| = \sum_{n=1}^{\infty} (-1)^{n+1} \cdot 2^n \cdot \frac{x^{n+1}}{n}$$

$$= 2 \cdot x^2 - \frac{2^2}{2} x^3 + \frac{2^3}{3} x^4 - \frac{2^4}{4} x^5 + \dots$$

$$\text{if } |-2x|<1 \iff -1 < 2x < 1 \iff -\frac{1}{2} < x < \frac{1}{2}$$

So, the Maclaurin series of f is $\sum_{n=1}^{\infty} (-1)^{n+1} \frac{2^n}{n} x^{n+1}$

if $-\frac{1}{2} < x < \frac{1}{2}$.

- If $x = -\frac{1}{2}$, then $\sum_{n=1}^{\infty} (-1)^{n+1} \cdot \frac{2^n}{n} \left(-\frac{1}{2}\right)^{n+1} = \sum_{n=1}^{\infty} \frac{1}{2n}$ is divergent

since it is harmonic.

- If $x = \frac{1}{2}$, then $\sum_{n=1}^{\infty} (-1)^{n+1} \cdot \frac{2^n}{n} \cdot \left(\frac{1}{2}\right)^{n+1} = \sum (-1)^{n+1} \cdot \frac{1}{2n}$ is

convergent by AST.

So, the interval of convergence is $\left(-\frac{1}{2}, \frac{1}{2}\right]$.

⑤ Use the Maclaurin series of $f(x) = \arctan x$ to find the sum S of the series

$$S = \sum_{n=0}^{\infty} \frac{(-1/3)^n}{2n+1}.$$

Soln:

Recall: The Maclaurin series of $f(x) = \arctan x$ is

$$\arctan x = \sum_{n=0}^{\infty} \frac{(-1)^n}{2n+1} x^{2n+1} \quad \text{if } -1 \leq x \leq 1.$$

$$S = \sum_{n=0}^{\infty} \frac{(-1)^n}{2n+1} \cdot \left(\frac{1}{3}\right)^n = \sum_{n=0}^{\infty} \frac{(-1)^n}{2n+1} \cdot \left(\frac{1}{\sqrt{3}}\right)^{2n}$$

$$= \sqrt{3} \cdot \sum_{n=0}^{\infty} \frac{(-1)^n}{2n+1} \left(\frac{1}{\sqrt{3}}\right)^{2n+1}$$

$$= \sqrt{3} \cdot \arctan\left(\frac{1}{\sqrt{3}}\right)$$

since $1/\sqrt{3} \in [-1,1]$

$$= \sqrt{3} \cdot \frac{\pi}{6}.$$

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