(1) Determine whether the following series converge absoluetly converge conditionally, or diveges.

(a) $\sum_{n=1}^{\infty} \frac{(-1)^n}{n^{3/2}}$

<u>Recall</u> (The alternating series test)

If a series $\sum_{n=1}^{\infty} a_n$ satisfies

(i) It is an alternating $(a_n \cdot a_{n+1} < 0 \quad \forall n)$

(ii) $|a_n|$ is decreasing

(iii) $\lim_{n\to\infty} |a_n| = 0$

then $\sum_{n=1}^{\infty} a_n$ is convergent

(Absolute Convergence test)

Let $\sum a_n$ be a series with non-positive terms

$\sum_{n=1}^{\infty} a_n$ is called absoluetly conv. if $\sum_{n=1}^{\infty} |a_n|$ is conv.

$\sum_{n=1}^{\infty} a_n$ is called conditionally conv. if $\sum_{n=1}^{\infty} a_n$ is conv.
but $\sum_{n=1}^{\infty} |a_n|$ is divergent

soln:

Let $a_n = \frac{(-1)^n}{n^{3/2}} \quad \forall n \geq 1.$

Then, $|a_n| = \frac{1}{n^{3/2}} \quad \forall n \geq 1$

Now consider $\sum_{n=1}^{\infty} |a_n| = \sum_{n=1}^{\infty} \frac{1}{n^{3/2}}$ which is convergent

by p-series test $(p = 3/2 > 1)$

So, $\sum_{n=1}^{\infty} (-1)^n \frac{1}{n^{3/2}}$ is absolutely convergent.

(b) $\sum_{n=1}^{\infty} \frac{(-1)^n}{n^{1/2}}$

soln: Say $a_n = \frac{(-1)^n}{n^{1/2}} \quad \forall n \geq 1$

Then $|a_n| = \frac{1}{n^{1/2}} \quad \forall n \geq 1.$

Now consider $\sum_{n=1}^{\infty} |a_n| = \sum_{n=1}^{\infty} \frac{1}{n^{1/2}}$ which is divergent

by p-series test $(p = 1/2 \leq 1)$.

(i) $a_n = \frac{(-1)^n}{n^{1/2}}$ is alternating since $a_n \cdot a_{n+1} < 0$

(ii) $|a_n| = \frac{1}{n^{1/2}}$ is decreasing:

$$|a_{n+1}| = \frac{1}{\sqrt{n+1}} < \frac{1}{\sqrt{n}} = |a_n|$$

(iii) $\lim_{n \to \infty} |a_n| = \lim_{n \to \infty} \frac{1}{\sqrt{n}} = 0.$

Then, $\sum_{n=1}^{\infty} a_n = \sum_{n=1}^{\infty} \frac{(-1)^n}{\sqrt{n}}$ is convergent by AST.

$\therefore \sum_{n=1}^{\infty} \frac{(-1)^n}{\sqrt{n}}$ is conditionally convegent.

c) $\displaystyle\sum_{n=1}^{\infty} \frac{\cos(n\pi)(n^3-3n^2+7)}{2n^3+13}$

Soln:

Say $a_n = \dfrac{\cos(n\pi)(n^3-3n^2+7)}{2n^3+13} = \dfrac{(-1)^n \cdot (n^3-3n^2+7)}{2n^3+13}$

, $n \geq 1$.

$$\left.\begin{aligned}\lim_{n\to\infty} a_{2n} &= \lim_{n\to\infty} \frac{n^3-3n+7}{2n^3+13} = \frac{1}{2}\\ \lim_{n\to\infty} a_{2n+1} &= \lim_{n\to\infty} \frac{(-1)(n^3-3n+7)}{2n^3+13} = -1/2\end{aligned}\right\} \neq$$

So, $\lim_{n\to\infty} a_n$ does not exist. By the "n-th term test",

$\sum_{n=1}^{\infty} a_n$ is divergent

! Be careful: $a_n = \dfrac{(-1)^n (n^3-3n^2+7)}{2n^3+13}$

$$\lim_{n\to\infty} |a_n| = \lim_{n\to\infty} \frac{n^3-3n^2+7}{2n^3+13} = \frac{1}{2} \neq 0$$

You cannot apply AST. AST does not say about the divergency of a series.

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(2) For the following series, find the smallest integer n that ensures that the partial sum S_n approximates the sum S of the series with $|\text{error}| = |S - S_n| < 0.001$

1a) $\sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{(2n)!}$

Soln :

Recall (Error estimate for altercating series)

If $\sum_{n=1}^{\infty} a_n$ converges where $\{a_n\}$ satisfies the conditions of the alternating series test, then

$$S = S_N + R_N = \underbrace{a_1 + a_2 + \cdots + a_N}_{S_N} + \underbrace{a_{N+1} + a_{N+2} + \cdots}_{R_N}$$

$$|S - S_N| = |R_N| = |\text{Error}| \leq |S_{N+1} - S_N| = |a_{N+1}|$$

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Soln:

Let $a_n = \frac{(-1)^{n-1}}{(2n)!}$ $\forall n \geq 1$

(i) a_n is alternating since $a_n \cdot a_{n+1} < 0$

(ii) $|a_n| = \frac{1}{(2n)!}$ is decreasing.

$$|a_{n+1}| = \frac{1}{(2n+2)!} = \frac{1}{2n!(2n+1)(2n+2)} < \frac{1}{2n!} = |a_n|$$

(iii) $\lim\limits_{n\to\infty} |a_n| = \lim\limits_{n\to\infty} \frac{1}{2n!} = 0.$

So, by AST $\sum\limits_{n=1}^{\infty} \frac{(-1)^{n-1}}{(2n)!}$ is convergent. Assume that it converges to S.

By the error estimite for alternating series,

$$\underset{|S-S_N|}{\overset{\parallel}{|Error|}} \leq |a_{N+1}| = \frac{1}{(2N+2)!} \leq 10^{-3} = \frac{1}{1000}$$

For $N=2$, $\frac{1}{6!} = \frac{1}{720} \approx 0{,}0014 > 0{,}001$

For $N=3$, $\frac{1}{8!} \approx 0{,}000025 < 0{,}001$

So, $|Error| < 0{,}001$ for all $N \geq 3$. $\therefore$ The smallest integer s.t. $|Error| < 0{,}001$ is $N=3$.

$$\sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{(2n)!} = \frac{1}{2} - \frac{1}{4!} + \frac{1}{6!} \underbrace{- \frac{1}{8!} + \frac{1}{10!} - \cdots}_{|\text{Error}| < 10^{-3}}$$

$$\sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{2n!} \approx \frac{1}{2} - \frac{1}{4!} + \frac{1}{6!} .$$

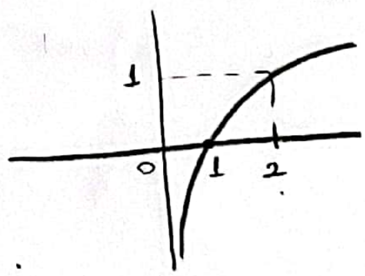
(b) $$\sum_{n=2}^{\infty} \frac{(-1)^n}{n \cdot \log_2(n)}$$

Soln:

Let $$a_n = \frac{(-1)^n}{n \cdot \log_2(n)} \quad \forall n \geq 1 .$$

(i) a_n is alternating since $a_n \cdot a_{n+1} < 0 \quad \forall n$

$y = \log_2 x.$



(ii) $$|a_n| = \frac{1}{n \cdot \underbrace{\log_2(n)}_{>0 \text{ if } n \geq 2}}$$ is decreasing.

$$|a_{n+1}| = \frac{1}{n+1 \cdot \log_2(n+1)} \underset{\uparrow}{<} \frac{1}{n \cdot \log_2 n} = |a_n| .$$

$n+1 > n$

$\log_2(n+1) > \log_2 n.$

(iii) $$\lim_{n \to \infty} |a_n| = \lim_{n \to \infty} \frac{1}{n \cdot \log_2(n)} = 0 .$$

So, by AST $\sum_{n=1}^{\infty} \frac{(-1)^n}{n \cdot \log_2(n)}$ is convegent.

By the error estimate for alternating series,

$$|\text{Error}| \le |a_{N+1}| = \frac{1}{(N+1)\log_2(N+1)} \leq 10^{-3} = \frac{1}{1000}$$

For $N = 139$, $|\text{Error}| \le \frac{1}{140 \cdot \log_2 140} \approx 0{,}00102 > 10^{-3}$

For $N = 140$, $|\text{Error}| \le \frac{1}{141 \cdot \log_2 141} \approx 0{,}00099 < 10^{-3}$

So, $|\text{Error}| < 10^{-3}$ for all $N \geq 140$

$\therefore$ The smallest integer s.t. $|\text{Error}| < 0{,}001$ is $N = 140$

③ For which values of x does the series

$$\sum_{n=1}^{\infty} \frac{(2\sin x)^n}{n} \quad \text{converges?}$$

Soln:

Let $a_n = \dfrac{(2\sin x)^n}{n}$ $\forall n \geq 1$ for some $x \in \mathbb{R}$.

$$\frac{a_{n+1}}{a_n} = \frac{(2\sin x)^{n+1}}{n+1} \cdot \frac{n}{(2\sin x)^n} = (2\sin x)\,\frac{n}{n+1}$$

$$\lim_{n\to\infty} \left| \frac{a_{n+1}}{a_n} \right| = \lim_{n\to\infty} |2\sin x| \cdot \Big|\underbrace{\frac{\cancel{n}}{\cancel{n}+1}}_{1}\Big| = |2\sin x| < 1$$

$\displaystyle\sum_{n=1}^{\infty} \frac{(2\sin x)^n}{n}$ is absolutely convergent, so convergent

if $|2\sin x| < 1$ by Ratio Test. That is,

$$-1 < 2\sin x < 1 \Rightarrow -\frac{1}{2} < \sin x < \frac{1}{2}$$

$$\Leftrightarrow -\frac{\pi}{6} < x < \frac{\pi}{6} \quad \text{or} \quad \frac{5\pi}{6} < x < \frac{7\pi}{6}$$

- $\boxed{x = -\frac{\pi}{6}} \Rightarrow \displaystyle\sum_{n=1}^{\infty} \frac{(-1)^n}{n}$ is $\boxed{\text{convergent}}$ by AST

- $\boxed{x = \frac{\pi}{6}} \Rightarrow \displaystyle\sum_{n=1}^{\infty} \frac{1}{n}$ $\boxed{\text{divergent}}$ (harmonic series)

- $\boxed{x = \frac{5\pi}{6}} \Rightarrow \sum_{n=1}^{\infty} \frac{1}{n}$ $\boxed{\text{divergent}}$ (harmonic)

- $\boxed{x = \frac{7\pi}{6}} \Rightarrow \sum_{n=1}^{\infty} \frac{(-1)^n}{n}$ $\boxed{\text{conv.}}$ by AST.

$\therefore \sum_{n=1}^{\infty} \frac{(2\sin x)^n}{n}$ is convergent if $x \in \left[-\frac{\pi}{6}+2\pi k, \frac{\pi}{6}+2\pi k\right) \cup$

$\left(\frac{5\pi}{6}+2\pi k, \frac{7\pi}{6}+2\pi k\right]$

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④ Determine the center, radius and interval of convergence of each of the following series.

(a) $\displaystyle\sum_{n=1}^{\infty} \frac{(2x+3)^n}{n^{120}}$

Recall (Power Series)

A formal series

$$a_0 + a_1(x-x_0) + a_2(x-x_0)^2 + \cdots = \sum_{n=0}^{\infty} a_n (x-x_0)^n$$

(for some fixed x_0) is called a power series.

The set of x's s.t. the power series converges is called the interval of convergence for the power series and x_0 is called the center of p.s.

[illegible]:

Soln:

Let $a_n = \dfrac{(2x+3)^n}{n^{120}} \quad \forall n \geq 1 \quad$ If $x \neq -3/2$,

$$\frac{a_{n+1}}{a_n} = \frac{\cancel{(2x+3)}^{n+1}}{(n+1)^{120}} \cdot \frac{n^{120}}{\cancel{(2x+3)}^{n}} = \frac{(2x+3)\, n^{120}}{(n+1)^{120}}$$

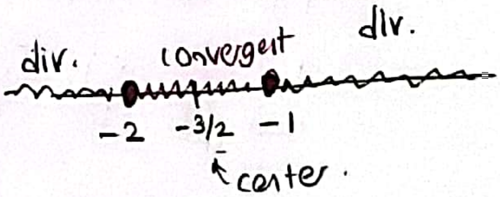
$$\lim_{n\to\infty} \left| \frac{a_{n+1}}{a_n} \right| = \lim_{n\to\infty} |2x+1| \cdot \underbrace{\left| \frac{n^{120}}{(n+1)^{120}} \right|}_{1} = |2x+3|$$

By the Ratio test, the series is abs. conv., so conv.

if $|2x+3| < 1 \iff -1 < 2x+3 < 1$

$\iff -4 < 2x < -2$

$\iff -2 < x < -1$



Therefore, the radius of convergence is $R = \frac{1}{2}$ with the center is $x_0 = -3/2$.

If $x_0 = -3/2$, then we have $\sum_{n=1}^{\infty} 0$ which is convergent

($S_n = \underbrace{0+0+\cdots+0}_{n\text{-times}}$. $\lim_{n\to\infty} S_n = 0 \Rightarrow \sum_{n=1}^{\infty} 0$ is convergent).

To determine the interval of convergence, we have to check the series for $x = -2$ and $x = -1$.

- If $x=-1$, then $a_n = \dfrac{(2.(-1)+3)^n}{n^{120}} = \dfrac{1}{n^{120}} \quad \forall n \geq 1$

Then, $\sum_{n=1}^{\infty} a_n = \sum_{n=1}^{\infty} \frac{1}{n^{120}}$ is convergent by p-series test $p = 120 > 1$.

- If $x=-2$, then $a_n = \dfrac{(2.(-2)+3)^n}{n^{120}} = \dfrac{(-1)^n}{n^{120}} \quad \forall n \geq 1.$

(I) a_n is alternating since $a_{n+1} \cdot a_n < 0$

(ii) $|a_n| = \dfrac{1}{n^{120}}$ is decreasing:

$$|a_{n+1}| = \frac{1}{(n+1)^{120}} < \frac{1}{n^{120}} = |a_n|.$$

(iii) $\lim_{n\to\infty} |a_n| = \lim_{n\to\infty} \frac{1}{n^{120}} = 0.$

By AST, $\sum_{n=1}^{\infty} a_n = \sum_{n=1}^{\infty} \frac{(-1)^n}{n^{120}}$ is convergent.

Thus, the interval of convergence is $[-2, -1]$.

b) $\sum_{n=1}^{\infty} \frac{e^n}{n!} (x-120)^n$

Soln :

Let $a_n = \frac{e^n}{n!} (x-120)^n$ for $n \geq 1$. If $x \neq 120$

$$\frac{a_{n+1}}{a_n} = \frac{\cancel{e^{n+1}} \cancel{(x-120)^{n+1}}}{\cancel{(n+1)!}} \cdot \frac{\cancel{n!}}{\cancel{e^n} \cdot \cancel{(x-120)^n}} = \frac{e \cdot (x-120)}{n+1}$$

$$\lim_{n\to\infty} \left|\frac{a_{n+1}}{a_n}\right| = \lim_{n\to\infty} \left|\frac{e}{n+1} (x-120)\right| = \lim_{n\to\infty} \left|\frac{e}{n+1}\right| \cdot |x-120| = 0 < 1$$

By the Ratio test, $\sum_{n=1}^{\infty} |a_n|$ is convegent if

$\lim_{n\to\infty} \left|\frac{a_{n+1}}{a_n}\right| = L < 1$. Since $L = 0 < 1$ for all x,

$\sum_{n=1}^{\infty} \frac{e^n}{n!} (x-120)^n$ is abs. conv., so convergent for all $x \in \mathbb{R} - \{120\}$

If $x = 120$, then we have $\sum_{n=1}^{\infty} 0$ which is convergent.

Thus, the intervall of convogence is $(-\infty, \infty)$ and

the radius of convergence is ∞

c) $\sum_{n=1}^{\infty} n^n . (x-2022)^n$

<u>Soln</u>:

Let $a_n = n^n . (x-2022)^n \quad \forall n \geq 1.$

$$|a_n|^{1/n} = |n.(x-2022)|^{n \cdot \frac{1}{n}} = |n|.|x-2022|$$

$$\lim_{n \to \infty} |a_n|^{1/n} = \lim_{n \to \infty} |n|.|x-2022| = \infty . > 1. \quad \text{if } x \neq 2022$$

By the Ratio test, $\sum_{n=1}^{\infty} |a_n|$ is divergent for all

$x \in \mathbb{R} - \{2022\}.$

If $x = 2022$, then we have $\sum_{n=1}^{\infty} 0$ which is convergent

Thus, the radius of convergence is zero and

the interval of convergence is $\{2022\}$

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9) $\sum_{n=1}^{\infty} \frac{\ln n}{\sqrt{n^5}}$

Soln:

Say $a_n = \frac{\ln n}{\sqrt{n^5}} > 0 \quad \forall n \geq 1$

$$a_n = \frac{\ln n}{\sqrt{n^5}} \underset{\uparrow}{\leq} \frac{n}{n^{5/2}} = \frac{1}{n^{3/2}} := b_n$$

$\ln x < x \quad \forall x \geq 1$ (Show! Exc)

Consider $\sum_{n=1}^{\infty} 1/n^{3/2}$ where $b_n = \frac{1}{n^{3/2}} > 0 \quad \forall n \geq 1.$

$\sum_{n=1}^{\infty} \frac{1}{n^{3/2}}$ is convergent by p-series test ($p = 3/2 > 1$)

By CT, $\sum_{n=1}^{\infty} \frac{\ln n}{\sqrt{n^5}}$ is also convergent.

h) $\sum_{n=0}^{\infty} \frac{n+2}{n!}$

Soln:

Recall (Ratio test)

Consider the series $\sum_{n=1}^{\infty} a_n$, $a_n > 0$ $\forall n$

(i) If $\lim_{n\to\infty} \left| \frac{a_{n+1}}{a_n} \right| < 1$, then $\sum_{n=1}^{\infty} a_n$ is convergent

(ii) If $\lim_{n\to\infty} \left| \frac{a_{n+1}}{a_n} \right| > 1$ OR $\lim_{n\to\infty} \left| \frac{a_{n+1}}{a_n} \right| = \infty$, then

$\sum_{n=1}^{\infty} a_n$ diverges.

(iii) If $\lim_{n\to\infty} \left| \frac{a_{n+1}}{a_n} \right| = 1$, then the test is inconclusive.

Say $a_n = \frac{n+2}{n!}$ $\forall n \geq 0$.

Then, $a_{n+1} = \frac{n+3}{(n+1)!} = \frac{n+3}{n!(n+1)}$

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$$\lim_{n\to\infty} \frac{a_{n+1}}{a_n} = \lim_{n\to\infty} \frac{n+3}{\cancel{n!}(n+1)} \cdot \frac{\cancel{n!}}{n+2}$$

$$= \lim_{n\to\infty} \frac{n+3}{(n+2)(n+1)} = 0 < 1$$

By Ratio test,

$\sum_{n=0}^{\infty} \frac{n+2}{n!}$ is convergent.

(18) $\sum_{n=1}^{\infty} \frac{n!}{2^{n^2}}$

Soln: Say $a_n = \frac{n!}{2^{n^2}} \quad \forall n \geq 1$

$$a_{n+1} = \frac{(n+1)!}{2^{(n+1)^2}} = \frac{n!(n+1)}{2^{n^2} \cdot 2^{2n+1}}$$

$$\lim_{n\to\infty} \frac{a_{n+1}}{a_n} = \lim_{n\to\infty} \frac{\cancel{n!}(n+1)}{\cancel{2^{n^2}} \cdot 2^{2n+1}} \cdot \frac{\cancel{2^{n^2}}}{\cancel{n!}} = \lim_{n\to\infty} \frac{n+1}{2^{2n+1}} = 0 < 1$$

Note: Define $f(x) = \frac{x+1}{2^{2x+1}}$ on $\mathbb{R}$ s.t. $f(n) = \frac{n+1}{2^{2n+1}} = b_n$

$$\lim_{x\to\infty} f(x) = \lim_{x\to\infty} \frac{x+1}{2^{2x+1}} \left[\frac{\infty}{\infty}\right] \overset{L'H}{=} \lim_{x\to\infty} \frac{1}{2 \cdot \ln 2 \cdot 2^{2x+1}} = 0.$$

So, $\sum_{n=1} \frac{n!}{2^{n^2}}$ is convergent

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j) $\sum_{n=1} \frac{2^n}{n^{\sqrt{n}}}$

Soln:

Recall (Root Test)

Consider the series $\sum_{n=1}^{\infty} a_n$, $a_n > 0$ $\forall n$

(i) If $\lim_{n\to\infty} \sqrt[n]{|a_n|} < 1$, then $\sum_{n=1}^{\infty} a_n$ is abs. conv. (convergent)

(ii) If $\lim_{n\to\infty} \sqrt[n]{|a_n|} > 1$ OR $\lim_{n\to\infty} \sqrt[n]{a_n} = \infty$, then

$\sum_{n=1}^{\infty} a_n$ is divergent.

(iii) If $\lim_{n\to\infty} \sqrt[n]{a_n} = 1$, then the test inconclusive.

Say $a_n = \frac{2^n}{n^{\sqrt{n}}} > 0$ $\forall n \geq 1$ $(|a_n| = a_n)$

$$a_n^{1/n} = \left(\frac{2^n}{n^{\sqrt{n}}}\right)^{1/n} = \frac{2}{n^{1/\sqrt{n}}}$$ Evaluate $\lim_{n\to\infty} \frac{2}{n^{1/\sqrt{n}}}$

Define $f(x) = x^{1/\sqrt{x}}$ on $(0,\infty)$ s.t. $f(n) = n^{1/\sqrt{n}}$ $\forall n \geq 1$

$$\lim_{x\to\infty} x^{1/\sqrt{x}} = ?$$

k) $\sum_{n=1}^{\infty} \frac{n!\,(n+1)!}{(3n)!}$

Soln:

Say $a_n = \frac{n!\,(n+1)!}{(3n)!} > 0 \quad \forall n \geq 1$

$$a_{n+1} = \frac{(n+1)!\,(n+2)!}{(3n+3)!}$$

$$\frac{a_{n+1}}{a_n} = \frac{\cancel{(n+1)!}\,\overset{(n+1)(n+2)}{\cancel{(n+2)!}}}{\underset{(3n+1)(3n+2)(3n+3)}{\cancel{(3n+3)!}}} \cdot \frac{\cancel{(3n)!}}{\cancel{n!}\,\cancel{(n+1)!}}$$

$$= \frac{n+2}{3\cdot(3n+1)(3n+2)}$$

$$\lim_{n\to\infty} \frac{a_{n+1}}{a_n} = \lim_{n\to\infty} \frac{n+2}{3(3n+1)(3n+2)} = 0 < 1$$

$\therefore \sum_{n=1}^{\infty} \frac{n!\,(n+1)!}{(3n)!}$ converges by Ratio Test.