MATH 120 - RECITATION PROBLEMS - WEEK 15

1. If C is the intersection of $z = \ln(1+x)$ and $y = x$ from $(0,0,0)$ to $(1,1,\ln 2)$ evaluate,

$$\int_C (2x\sin(\pi y) - e^z)\,dx + (\pi x^2 \cos \pi y - 3e^z)\,dy - xe^z\,dz.$$

Sol^n: Firstly, let us check whether F is conservative or not where $F = \langle \underbrace{2x\sin(\pi y) - e^z}_{F_1}, \underbrace{\pi x^2 \cos \pi y - 3e^z}_{F_2}, \underbrace{-xe^z}_{F_3} \rangle$

$$\frac{\partial F_1}{\partial y} = 2\pi x \cos(\pi y) \overset{?}{=} \frac{\partial F_2}{\partial x} = 2\pi x \cos \pi y \quad \checkmark$$

$$\frac{\partial F_1}{\partial z} = -e^z \overset{?}{=} \frac{\partial F_3}{\partial x} = -e^z \quad \checkmark$$

$$\frac{\partial F_2}{\partial z} = -3e^z \overset{?}{\neq} \frac{\partial F_3}{\partial y} = 0 \quad X$$

$\Rightarrow F$ is not conservative.

Modify F so that it becomes conservative.

$$\int_C F\cdot dr = \int_C (2x\sin(\pi y)-e^z)\,dx + (\pi x^2\cos(\pi y)-3e^z)\,dy + (-xe^z-3ye^z)\,dz + \int_C 3ye^z\,dz$$

$$= \int_C \underbrace{(2x\sin(\pi y)-e^z)\,dx + (\pi x^2\cos(\pi y)-3e^z)\,dy + (-xe^z-3ye^z)\,dz}_{\vec{G}(x,y,z)\cdot d\vec{r}}$$

$$+ \int_C 3ye^z\,dz = \int_C \vec{F}\cdot d\vec{r}$$

Note that G satisfies the necessary cond., so it can be conservative. Assume that ϕ is a potential function

$$G = \vec{\nabla}\phi$$

$$\frac{\partial \phi}{\partial x} = 2x\sin(\pi y) - e^z, \quad \frac{\partial \phi}{\partial y} = \pi x^2\cos(\pi y) - 3e^z$$

$$\frac{\partial \phi}{\partial z} = -xe^z - 3ye^z$$

$$\phi(x,y,z) = \int (2x\sin(\pi y) - e^z)\,dx = x^2\sin(\pi y) - xe^z + g(y,z)$$

$$\frac{\partial\phi}{\partial y} = \pi x^2\cos(\pi y) + g_y(y,z) = \pi x^2\cos(\pi y) - 3e^z$$

$$\Rightarrow g_y(y,z) = -3e^z \quad \Rightarrow g(y,z) = -3ye^z + f(z)$$

$$\Rightarrow \phi(x,y,z) = x^2\sin(\pi y) - xe^z - 3ye^z + f(z)$$

$$\frac{\partial\phi}{\partial z} = -xe^z - 3ye^z + f'(z) = -xe^z - 3ye^z$$

$\Rightarrow f'(z) = 0 \quad \Rightarrow f(z) = c$ for some $c \in \mathbb{R}$

Take $c = 0$, $\phi(x,y,z) = x^2\sin(\pi y) - xe^z - 3ye^z$ is

a potential function. By independence of path, we have

$$\int_C \vec{G}\cdot d\vec{r} = \phi(1,1,\ln 2) - \phi(0,0,0) = -8$$

$$\int_C \vec{F}\cdot d\vec{r} = \int_C \vec{G}\cdot d\vec{r} + \int_C 3ye^z\,dz$$

$$= -8 + \int_0^1 3te^{\cancel{\ln(1+t)}}\cdot\cancel{\frac{1}{1+t}}\,dt = -8 + \frac{3t^2}{2}\Big|_0^1$$

$$= -8 + \frac{3}{2} = -\frac{13}{2}$$

$x = y = t$

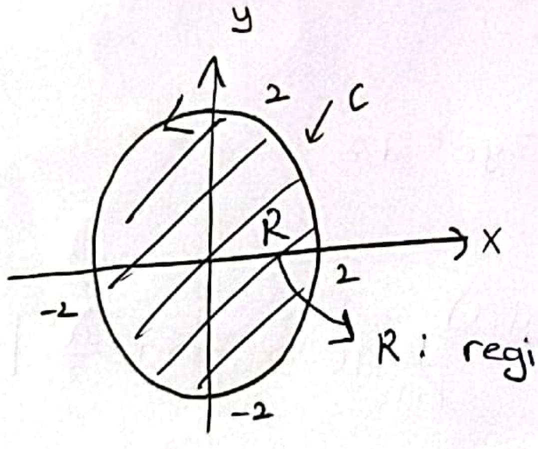
$z = \ln(1+t) \Rightarrow dz = \frac{1}{1+t}\,dt$, $t \in [0,1]$

(2.) Evaluate $\oint (3x+4y)\,dx + (2x+3y^2)\,dy$ around $x^2+y^2=4$.

<u>soln</u>:

Green's Thm: Let R be a regular, closed region in the xy-plane whose boundary C, consists of one or more piecewise smooth simple closed curves that are positively oriented with respect to R. If $F = F_1(x,y)\,\vec{i} + F_2(x,y)\,\vec{j}$ is a smooth v.f. on R, then

$$\oint_C F_1(x,y)\,dx + F_2(x,y)\,dy = \iint_R \left(\frac{\partial F_2}{\partial x} - \frac{\partial F_1}{\partial y}\right) dA.$$



- R is a regular closed region
- $\partial R = C$ is a smooth simple closed curve, positively oriented in $\mathbb{R}^2$

R : region enclosed by C

- $F(x,y) = \langle \underbrace{(3x+4y)}_{F_1(x,y)}, \underbrace{2x+3y^2}_{F_2(x,y)} \rangle$

By Green's thm,

$$\oint_C \vec{F}\cdot d\vec{r} = \iint_R \left(\frac{\partial F_2}{\partial x} - \frac{\partial F_1}{\partial y}\right) dA = \iint_R (2-4)\,dA = -2\underbrace{\iint_R 1.\,dA}_{\text{Area}(R)}$$

$$= -2 \cdot \pi \cdot 2^2 = -8\pi.$$

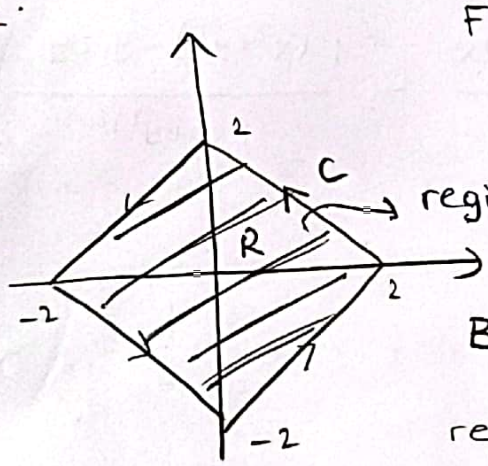
③ Given the vector field

$$F = \frac{y\vec{i} - x\vec{j}}{x^2+y^2}$$

compute $\oint_C \vec{F}\cdot d\vec{r}$ where C is the curve enclosing the square with vertices $(2,0), (0,2), (-2,0), (0,-2)$ oriented in the positive direction.

<u>soln</u>:

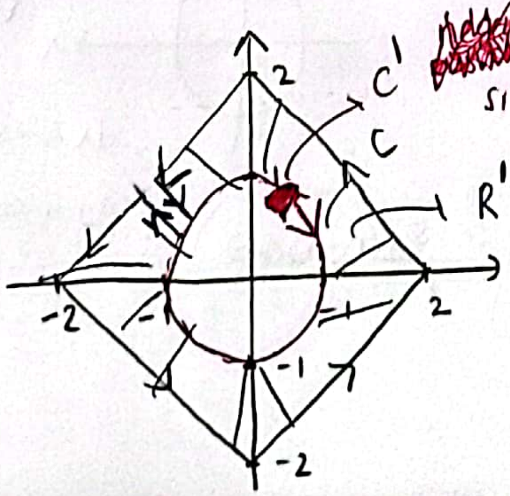
$$F(x,y) = \underbrace{\frac{y}{x^2+y^2}}_{F_1}\,\vec{i} + \underbrace{\frac{-x}{x^2+y^2}}_{F_2}\,\vec{j}$$



→ region enclosed by C

Be careful! It is not a regular region. F_1 and F_2 are not defined at $(0,0) \in R$. So, we can not use Green's theorem.

Consider the curve C': $x^2+y^2=1$ (inside the region R).



simple closed

→ R' region enclosed by $C \cup C'$

- R' = regular closed region
- $\partial R' = C \cup C'$ = piecewise smooth simple closed curves with positively oriented

By Green's theorem,

$$\oint_{C \cup C'} \vec{F} \cdot dr = \iint_R \left(\frac{\partial F_2}{\partial x} - \frac{\partial F_1}{\partial y} \right) dA$$

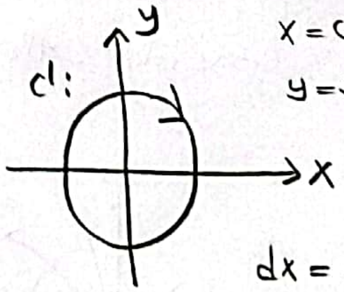
$$= \iint_R \left(\frac{-1(x^2+y^2)+x \cdot 2x}{(x^2+y^2)^2} - \frac{1 \cdot (x^2+y^2) - y \cdot 2y}{(x^2+y^2)^2} \right) dA$$

$$= \iint_R 0 \cdot dA = 0.$$

$$\oint_{C \cup C'} \vec{F} \cdot d\vec{r} = \oint_C \vec{F} \cdot d\vec{r} + \int_{C'} \vec{F} \cdot d\vec{r} = 0$$

$$\Rightarrow \oint_C \vec{F} \cdot d\vec{r} = - \int_{C'} \vec{F} \cdot d\vec{r}$$

C':



$x = \cos\theta$

$y = \sin\theta$

$\theta \in 2\pi \to 0$

$dx = -\sin\theta \, d\theta$

$dy = \cos\theta \, d\theta$

$$\oint_C \vec{F} d\vec{r} = -\left(-\int_0^{2\pi} \left(\frac{\sin\theta}{1} (-\sin\theta) d\theta + \frac{\cos\theta}{1} \cos\theta \, d\theta \right) \right)$$

$$= -\int_0^{2\pi} 1 \cdot d\theta = -2\pi.$$

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④ Find the area enclosed by the curve

$$\vec{r}(t) = \left\langle \frac{2\cos t - \sin t}{2}, \sin t \right\rangle, \quad t \in [0, 2\pi]$$

Hint: Apply Green's theorem for the vector field

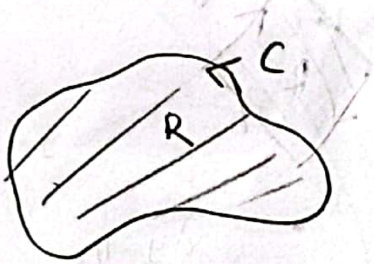
$F = \frac{1}{2}(-y\vec{i} + x\vec{j})$ along the curve $\vec{r}(t)$.

Soln:

$$x(t) = \frac{2\cos t - \sin t}{2}, \quad y(t) = \sin t, \quad t \in [0, 2\pi]$$

$$dx = \left(-\sin t - \frac{\cos t}{2}\right) dt, \quad dy = \cos t\, dt$$

$$\text{Area}(R) = \frac{1}{2} \int_C x\,dy - y\,dx$$



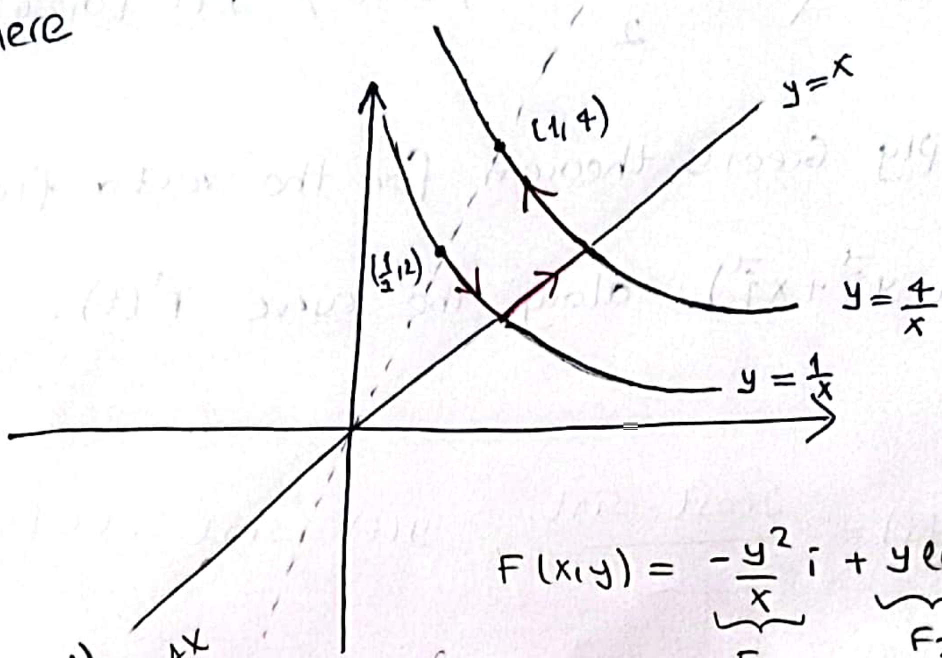
$$= \frac{1}{2} \int_0^{2\pi} \left(\cos t - \frac{\sin t}{2}\right) \cos t\, dt - \sin t \left(-\sin t - \frac{\cos t}{2}\right) dt$$

$$= \frac{1}{2} \int_0^{2\pi} \left(\cos^2 t - \cancel{\frac{\cos t \sin t}{2}} + \sin^2 t - \cancel{\frac{\sin t \cos t}{2}}\right) dt$$

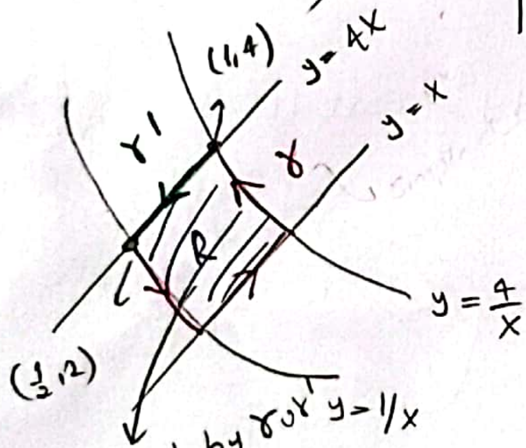
$$= \frac{1}{2} \int_0^{2\pi} 1 \cdot dt = \frac{1}{2} \cdot 2\pi = \pi.$$

(5) Evaluate $\displaystyle\int_{(\frac{1}{2},2)}^{(1,4)} -\frac{y^2}{x}\,dx + y\ln x\,dy$ along the curve

γ where



$$F(x,y) = \underbrace{-\frac{y^2}{x}}_{F_1}\,i + \underbrace{y\ln x}_{F_2}\,j$$

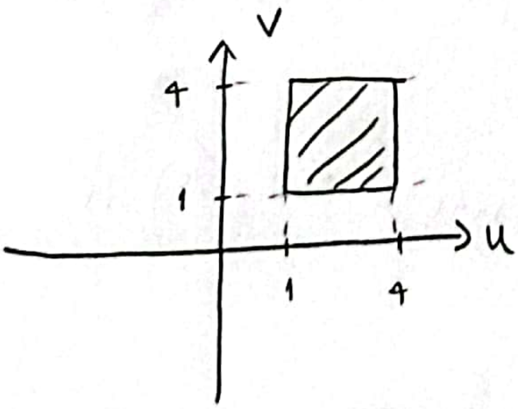


R enclosed by $\gamma \cup \gamma'$ y=1/x

Let $xy = u$, $\frac{y}{x} = v$

$$\frac{\partial(u,v)}{\partial(x,y)} = \begin{vmatrix} y & x \\ -\frac{y}{x^2} & 1/x \end{vmatrix} = \frac{2y}{x}$$

$$dA_{(x,y)} = \frac{1}{\frac{\partial(u,v)}{\partial(x,y)}}\,du\,dv$$



- R regular closed region
- $\partial R = \gamma \cup \gamma'$ piecewise smooth, closed curves with positively oriented

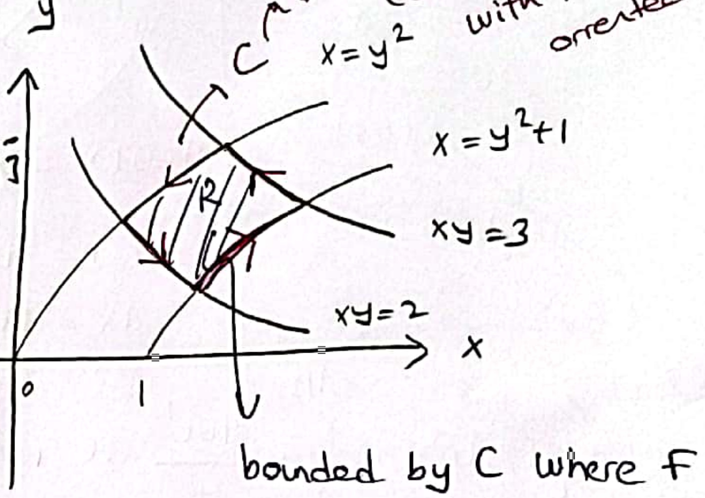
By Green's Thm,

$$\oint_{\gamma\cup\gamma'} \vec{F}\cdot d\vec{r} = \iint_R \left(\frac{\partial F_2}{\partial x} - \frac{\partial F_1}{\partial y}\right) dA = \iint_R \frac{1}{x} - \left(-\frac{2y}{x}\right) dA = \iint_R 3\frac{y}{x}\,dA$$

(6) Let C be the boundary of the region bounded by $x = y^2$, $x = 1 + y^2$, $xy = 2$, $xy = 3$ which is oriented counterclockwise. Evaluate $\oint_C (e^{x^3} - \frac{2}{3}y^3)\,dx + (\sin y^3 + \frac{1}{2}x^2)\,dy$

Soln:

$$F(x,y) = \underbrace{(e^{x^3} - \tfrac{2}{3}y^3)}_{F_1}\vec{i} + \underbrace{(\sin y^3 + \tfrac{x^2}{2})}_{F_2}\vec{j}$$



bounded by C where F is regular

By Green's theorem,

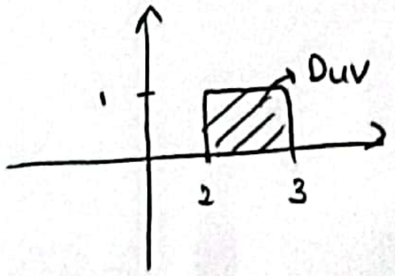
$$\oint_C \vec{F}\cdot d\vec{r} = \iint_R \left(\frac{\partial F_2}{\partial x} - \frac{\partial F_1}{\partial y}\right) dA = \iint_R (x - (-2y^2))\,dA$$

$$= \iint_R (x + 2y^2)\,dA$$

Let $u = xy$ and $v = x - y^2$

$$\frac{\partial(u,v)}{\partial(x,y)} = \begin{vmatrix} y & x \\ 1 & -2y \end{vmatrix} = -2y^2 - x \qquad dA = \left|\frac{\partial(x,y)}{\partial(u,v)}\right| du\,dv$$

$$= \frac{1}{x+2y^2}\,du\,dv$$



$$\oint_C F\cdot dr = \iint_{R_{uv}} x + 2y^2 \cdot \frac{1}{x+2y^2}\,du\,dv$$

$$= \iint_{R_{uv}} 1 \cdot du \cdot dv = 1.$$

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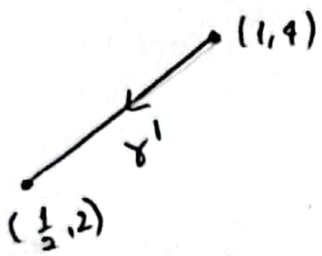
$$= \iint_R \frac{3y}{x}\, dA = \int_1^4\!\!\int_1^4 3\cancel{y}\,\frac{1}{2\cancel{y}}\,du\,dv = \frac{3}{2}\cdot 3\cdot 3 = \frac{27}{2}$$

$$\Rightarrow \int_{\gamma\cup\gamma'} F\cdot dr = \int_{\gamma} F\cdot dr + \int_{\gamma'} F\cdot dr = \frac{27}{2}$$



$y = 4x \Rightarrow$ Say $x(t) = t$ $\quad y(t) = 4t$

$t: 1 \to 1/2$

$dx = dt \quad dy = 4dt$

$$\int_{\gamma'} F\cdot d\vec{r} = \int_1^{1/2} -\frac{16t^2}{t}\,dt + 4t\ln t\,(4dt)$$

$$= \int_1^{1/2} (-16t + 16t\ln t)\,dt$$

$\downarrow$ By IBP $\quad u = \ln t \quad dv = t\,dt$

$$= -16\frac{t^2}{2} + 4t^2(2\ln t - 1)\Big|_1^{1/2}$$

$$= 9 - \ln 2.$$

$$\Rightarrow \int_{\gamma} \vec{F}\cdot d\vec{r} = \frac{27}{2} - 9 + \ln 2 = 9 + \ln 2.$$