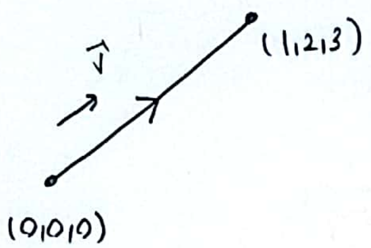
MATH 120 - 2021 - 2 - RECITATION PROBLEMS - WEEK 14

(1) 1a) Evaluate $\int_C x e^{yz} \, ds$ where C is the line segment from (0,0,0) to (1,2,3).

Soln:

$\vec{v} = \langle 1,2,3 \rangle$




$r(t) = \langle 0,0,0 \rangle + t \cdot \langle 1,2,3 \rangle$

$= \langle \underbrace{t}_{x(t)}, \underbrace{2t}_{y(t)}, \underbrace{3t}_{z(t)} \rangle \quad , \quad t \in [0,1]$

$r'(t) = \langle 1,2,3 \rangle \quad , \quad |r'(t)| = \sqrt{14}$

$ds = |r'(t)| \, dt$

$$\int_C x e^{yz} \, ds = \int_0^1 t e^{6t^2} \sqrt{14} \, dt = \sqrt{14} \int_0^6 e^u \frac{du}{12} = \frac{\sqrt{14}}{12} e^u \Big|_0^6$$

$6t^2 = u$

$12t \, dt = du$

$t = 0 \Rightarrow u = 0$

$t = 1 \Rightarrow u = 6$

$$= \frac{\sqrt{14}}{12} (e^6 - 1)$$

b) Evaluate $\int_C \sqrt{x^2+y^2}\, ds$ along the curve $C$ given by

$r(t) = 4\cos t\, \hat{i} + 4\sin t\, \hat{j} + 3t\, k \; , -2\pi \leq t \leq 2\pi .$

Soln:

$C: \quad r(t) = \langle \underbrace{4\cos t}_{x(t)}, \underbrace{4\sin t}_{y(t)}, \underbrace{3t}_{z(t)} \rangle \; , \quad t \in [-2\pi, 2\pi]$

$$r'(t) = \langle -4\sin t, 4\cos t, 3 \rangle$$

$$|r'(t)| = \sqrt{16\sin^2 t + 16\cos^2 t + 9}$$

$$= \sqrt{25} = 5$$

$$ds = |r'(t)|\, dt \Rightarrow ds = 5dt$$

$$\int_C \sqrt{x^2+y^2}\, ds = \int_{-2\pi}^{2\pi} \sqrt{(4\cos t)^2 + (4\sin t)^2}\, (5dt)$$

$$= \int_{-2\pi}^{2\pi} 20\, dt = 20 . (2\pi - (-2\pi))$$

$$= 20 . 4\pi = 80\pi .$$

c) Find $\int_C \sqrt{1+4x^2z^2}\, ds$ where $C$ is the curve of the intersection of the surfaces $x^2+z^2=1$ and $y=x^2$.

Soln:

$$x(t)=\cos t,\quad y(t)=\cos^2(t),\quad z(t)=\sin(t),\quad t\in[0,2\pi]$$

$$C: r(t) = \langle \cos(t), \cos^2(t), \sin(t)\rangle,\quad t\in[0,2\pi]$$

$$r'(t) = \langle -\sin t, -2\cos t\sin t, \cos t\rangle$$

$$|r'(t)| = \sqrt{\sin^2 t + 4\cos^2 t\sin^2 t + \cos^2 t} = \sqrt{1+4\cos^2\sin^2 t}$$

$$\int_C \sqrt{1+4x^2z^2}\,ds = \int_0^{2\pi} \sqrt{1+4\cos^2 t\sin^2 t}\cdot\sqrt{1+4\cos^2 t\sin^2 t}\,dt$$

$$= \int_0^{2\pi} (1+4\cos^2 t\sin^2 t)\,dt$$

$$= \int_0^{2\pi} dt + \int_0^{2\pi} 4\cos^2 t\sin^2 t\,dt \quad \downarrow \underline{\underline{\text{Exc}}}$$

$$= 2\pi + \left(-\frac{\sin(4t)-4t}{8}\Big|_0^{2\pi}\right)$$

$$= 2\pi - \frac{\sin(8\pi)}{8} + \frac{8\pi}{8} + \frac{\sin 0}{8} - 0$$

$$= 3\pi$$

(2) Describe the parametric curve C given by

$x = a\cos t \sin t$ , $y = a\sin 2t$ , $z = bt$

Express (do NOT evaluate) the length of C between

$t=0$ and $t=T>0$.

Soln:

$$x = \frac{a\sin 2t}{2} \Rightarrow y = 2x$$

Length of the curve $= \int_C ds = \int_a^b |r'(t)|\,dt$

between a and b.

$$r(t) = \left\langle \frac{a\sin(2t)}{2},\ a\sin(2t),\ bt \right\rangle$$

$$r'(t) = \langle a\cos(2t),\ 2a\cos(2t),\ b \rangle$$

Length of curve $= \int_0^T |r'(t)|\,dt = \int_0^T \sqrt{(a\cos(2t))^2 + (2a\cos(2t))^2 + b^2}\,dt$

$$= \int_0^T \sqrt{5a^2\cos^2 2t + b^2}\,dt .$$

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③ Find the total length of the curve of the intersection of the sphere $x^2+y^2+z^2=1$ and the elliptic cylinder $x^2+2z^2=1$.

<u>soln:</u>

Because of symmetry, we have 8 segments on each octant. To find total length, we will calculate one length and multiply with 8 on the intersection.

Let $x=\cos t$, $z=\frac{\sin t}{\sqrt{2}}$ (defines ellipse)

$$\cos^2 t+y^2+\frac{\sin^2 t}{2}=1 \Rightarrow y^2=1-\cos^2 t-\frac{\sin^2 t}{2}$$

$$=\sin^2 t-\frac{\sin^2 t}{2}$$

$$=\frac{\sin^2 t}{2}$$

$$\Rightarrow y=\frac{\sin t}{\sqrt{2}} \text{ or } -\frac{\sin t}{\sqrt{2}}$$

Parametrization for the line segment in $1^{st}$ octant is

$$r(t)=\left\langle \cos(t), \frac{\sin t}{\sqrt{2}}, \frac{\sin t}{\sqrt{2}} \right\rangle, \quad t\in\left[0,\frac{\pi}{2}\right]$$

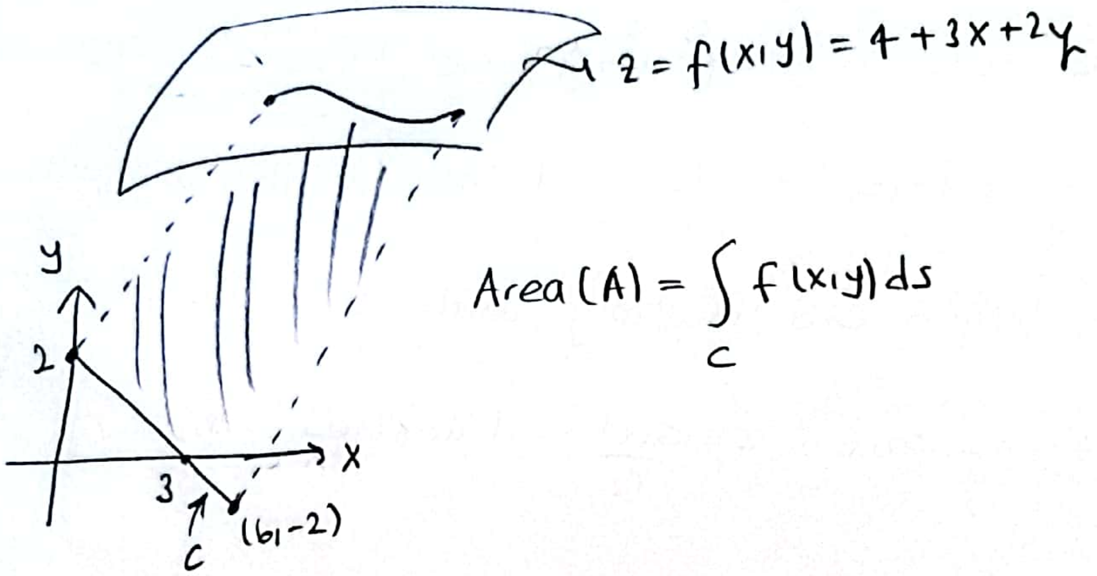
$$r'(t)=\left\langle -\sin t, \frac{\cos t}{\sqrt{2}}, \frac{\cos t}{\sqrt{2}} \right\rangle$$

$$|r'(t)|=\sqrt{\sin^2 t+\frac{\cos^2 t}{2}+\frac{\cos^2 t}{2}}=1$$

$$L=8\int^{\pi/2} 1\cdot dt=4\pi.$$

④ Find the area of one side of "wall" standing orthogonally on the curve $2x+3y=6$, $0 \le x \le 6$ and beneath the curve on the surface $f(x,y) = 4+3x+2y$.

<u>soln</u>:




$$\text{Area}(A) = \int_C f(x,y)\,ds$$

Parametrization of C: $x = t$, $y = \frac{6-2t}{3}$, $t \in [0,6]$

$$\Rightarrow c(t) = \langle t, \frac{6-2t}{3} \rangle, t \in [0,6]$$

$$c'(t) = \langle 1, \frac{-2}{3} \rangle, \quad \|c'(t)\| = \sqrt{1+\frac{4}{9}} = \frac{\sqrt{13}}{3}$$

$$\Rightarrow \text{Area}(A) = \int_C f(x,y)\,ds = \int_0^6 \left[4+3t+2\left(\frac{6-2t}{3}\right)\right]\frac{\sqrt{13}}{3}\,dt$$

$$= \frac{\sqrt{13}}{3}\left[\cancel{4t}+\frac{3t^2}{2}-\cancel{4t}+\frac{4}{3}\frac{t^2}{2}\right]\Big|_0^6$$

$$= \frac{\sqrt{13}}{3} \cdot 76 = 26\sqrt{13}.$$

⑤ (a) Given the vector field $F(x,y) = 2xy\,\bar{i} + (x^2+y^2)\,\bar{j}$,

compute the line integral $\int_C F \cdot dr$ where $C$ is the portion of the curve $\sqrt{x} + xy + \sqrt{y} = 7$ starting at $(4,1)$ and ending at $(1,4)$.

<u>Soln</u>:

First, we'll show that $\vec{F}(x,y)$ is conservative.

A vector field $\vec{F}$ is conservative if $\vec{F} = \vec{\nabla}\phi$ for some $\phi$, called potential function.

$$\frac{\partial \phi}{\partial x} = 2xy \quad \text{and} \quad \frac{\partial \phi}{\partial y} = x^2+y^2$$

$$\phi(x,y) = x^2y + g(y).$$

$$\frac{\partial \phi}{\partial y} = x^2 + g'(y) = x^2+y^2 \Rightarrow g'(y) = y^2 \Rightarrow g(y) = \frac{y^3}{3} + c,$$

$c \in \mathbb{R}$.

Take $c = 0$.

$\phi(x,y) = x^2y + \frac{y^3}{3}$ is a potential function for $F$.

So, $F$ is conservative. Then,

$$\int_C \vec{F}\, d\vec{r} = \phi(1,4) - \phi(4,1)$$

$$= 4 + \frac{4^3}{3} - 4^2 - \frac{1}{3} = \frac{63}{3} - 12 = \frac{27}{3} = 9.$$

b) Given the vector field $F(x,y,z) = (2xy+z)\,i + x^2 j + (x+2z)k$

compute the line integral $\int_C F \cdot dr$ where $C$ is the ~~portion~~ curve

from $(1,1,0)$ to $(0,0,3)$ that lies on the intersection

of the surfaces $2x+y+z=3$ and $9x^2+9y^2+2z^2=18$ in

the first octant.

Soln: A vector field $\vec{F}$ is conservative if $\vec{F} = \nabla \vec{\phi}$

$$\frac{\partial \phi}{\partial x} = 2xy+z \quad , \quad \frac{\partial \phi}{\partial y} = x^2 \quad , \quad \frac{\partial \phi}{\partial z} = x+2z$$

$$\phi(x,y,z) = x^2 y + zx + g(z,y)$$

$$\frac{\partial \phi}{\partial y} = x^2 + g_y(z,y) = x^2 \Rightarrow g_y(z,y) = 0$$

$$\Rightarrow g(z,y) = f(z)$$

$$\phi(x,y,z) = x^2 y + zx + f(z) \Rightarrow \frac{\partial \phi}{\partial z} = x + f'(z) = x + 2z$$

$$f'(z) = 2z \Rightarrow f(z) = z^2 + c,$$

$c \in \mathbb{R}$.

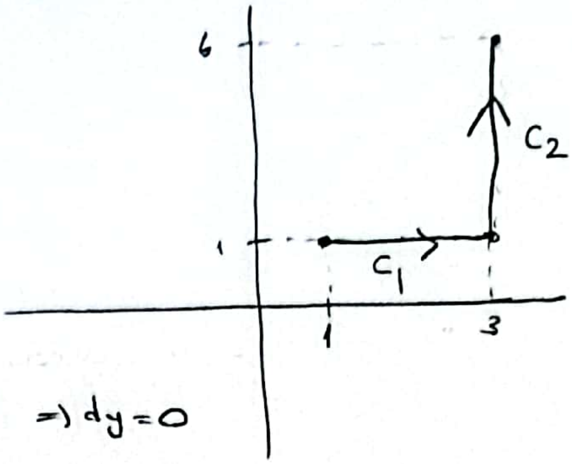
Take $c=0$, $\phi(x,y,z) = x^2 y + zx + z^2$ is a potential function.

So, $\vec{F}$ is conservative. Then,

$$\int_C \vec{F} \cdot d\vec{r} = \phi(0,0,3) - \phi(1,1,0) = 9 - 1 = 8.$$

⑥ Evaluate $\int_C \frac{1}{xy}\,dx + \frac{1}{x+y}\,dy$ along the path from $(1,1)$ to $(3,1)$ to $(3,6)$ using straight line segments.

soln :




$C_1(t) = \langle 1,1\rangle + t\langle 2,0\rangle$

$= \langle 1+2t, 1\rangle$, $t \in [0,1]$

$x(t) = 1+2t \Rightarrow dx = 2dt$ $\quad y(t) = 1 \Rightarrow dy = 0$

$C_2(t) = \langle 3,1\rangle + t\langle 0,5\rangle$

$= \langle 3, 1+5t\rangle$, $t \in [0,1]$.

$x(t) = 3 \Rightarrow dx = 0$ $\quad y(t) = 1+5t \Rightarrow dy = 5dt$

$$\underset{\substack{\shortparallel \\ C_1+C_2}}{\int_C} \frac{1}{xy}\,dx + \frac{1}{x+y}\,dy = \int_{C_1} \frac{1}{xy}\,dx + \frac{1}{x+y}\,dy + \int_{C_2} \frac{1}{xy}\,dx + \frac{1}{x+y}\,dy$$

$$= \int_0^1 \frac{1}{(1+2t)}\,2dt + \int_0^1 \frac{1}{4+5t}\,5dt$$

$$= \ln|1+2t|\Big|_0^1 + \ln|4+5t|\Big|_0^1 = \ln 3 - \ln 1 + \ln|9| - \ln|4|$$

$$= \ln 3 + \ln 9 - \ln 4$$

Note : $\frac{\partial}{\partial x}\left(\frac{1}{x+y}\right) \neq \frac{\partial}{\partial y}\left(\frac{1}{xy}\right) \Rightarrow F(x,y) = \left\langle \frac{1}{xy}, \frac{1}{x+y}\right\rangle$

is not conservative!