b) $\sum_{n=1}^{\infty} \sin^{2n}\theta$ for $0 < \theta < \frac{\pi}{2}$

Soln:

Say $a_n = \sin^{2n}\theta \quad \forall n \geq 1$ for $\theta \in (0, \frac{\pi}{2})$

$$\underset{(\sin^2\theta)^n}{\overset{\Vert}{}}$$

$$\sum_{n=0}^{\infty} (\sin^2\theta)^n = 1 + \sum_{n=1}^{\infty} (\sin^2\theta)^n$$

is a geometric series where $a = 1$ and $r = \sin^2\theta$.

$\sum_{n=0}^{\infty} (\sin^2\theta)^n$ is convergent and converges

to $\frac{1}{1-\sin^2\theta} = \frac{1}{\cos^2\theta}$ since $|r| = |\sin^2\theta| < 1$

(since $\theta \in (0, \frac{\pi}{2})$)

Then, we have

$$\frac{1}{\cos^2\theta} = 1 + \sum_{n=1}^{\infty} (\sin^2\theta)^n$$

$$\Rightarrow \sum_{n=1}^{\infty} (\sin^2\theta)^n = \frac{1}{\cos^2\theta} - 1 = \frac{1-\cos^2\theta}{\cos^2\theta} = \frac{\sin^2\theta}{\cos^2\theta} = \tan^2\theta.$$

c) $\sum_{n=0}^{\infty} \frac{2^{2n+1}}{\pi^{n-2}}$

Soln:

Say $a_n = \frac{2^{2n+1}}{\pi^{n-2}} \quad \forall n \geq 0$

$$\sum_{n=0}^{\infty} \frac{2^{2n+1}}{\pi^{n-2}} = \sum_{n=0}^{\infty} \frac{4^n \cdot 2}{\pi^{n-2}} = \sum_{n=0}^{\infty} \left(\frac{4}{\pi}\right)^{n-2} \cdot 4^2 \cdot 2$$

$$= \sum_{n=0}^{\infty} \left(\frac{4}{\pi}\right)^{n-1} \cdot \frac{\pi}{4} \cdot 4^2 \cdot 2$$

$$= \sum_{n=0}^{\infty} \left(\frac{4}{\pi}\right)^{n-1} \cdot 8\pi$$

$$= 2\pi^2 + \underbrace{\sum_{n=1}^{\infty} \left(\frac{4}{\pi}\right)^{n-1} \cdot 8\pi}_{\text{diverges}}$$

$\sum_{n=1}^{\infty} \left(\frac{4}{\pi}\right)^{n-1} \cdot 8\pi$ diverges since $|r| = \frac{4}{\pi} \geq 1$

$\therefore \sum_{n=0}^{\infty} \frac{2^{2n+1}}{\pi^{n-2}}$ is divergent

b) $\sum_{n=0}^{\infty} \left(\frac{n+1}{n+2}\right)^n$

Soln:

Say $a_n = \left(\frac{n+1}{n+2}\right)^n \quad \forall n > 0$

$$\lim_{n\to\infty} \left(\frac{n+1}{n+2}\right)^n = ?$$

Let $f(n) = a_n$ where $f(x) = \left(\frac{x+1}{x+2}\right)^x$

$$\lim_{x\to\infty} \left(\frac{x+1}{x+2}\right)^x = L$$

Take "ln" both sides:

$$\ln\left(\lim_{x\to\infty} \left(\frac{x+1}{x+2}\right)^x\right) = \ln L$$

$$\Rightarrow \ln L = \lim_{x\to\infty} \ln\left(\frac{x+1}{x+2}\right)^x = \lim_{x\to\infty} x \cdot \ln\left(\frac{x+1}{x+2}\right) \quad [\infty \cdot 0]$$

"ln" is cont.

$$= \lim_{x\to\infty} \frac{\ln\left(\frac{x+1}{x+2}\right)}{1/x} \quad \left[\frac{\infty}{\infty}\right]$$

$$= \lim_{x\to\infty} \frac{x+2-(x+1)}{(x+2)^2} \cdot \frac{x+2}{x+1} \cdot (-x^2)$$

$$= \lim_{x\to\infty} \frac{-x^2}{(x+2)(x+1)} = -1$$

$\Rightarrow \ln L = -1 \Rightarrow L = e^{-1} = \frac{1}{e}$ So, $\lim_{x\to\infty} \left(\frac{x+1}{x+2}\right)^x = \frac{1}{e}$

Since $f(n) = a_n$ and $\lim_{x \to \infty} \left(\frac{x+1}{x+2}\right)^x = \frac{1}{e}$,

$$\lim_{n \to \infty} \left(\frac{n+1}{n+2}\right)^n = \frac{1}{e} \neq 0$$

By the n-th term test, $\sum_{n=1}^{\infty} \left(\frac{n+1}{n+2}\right)^n$ is divergent.

c) $\sum_{n=10}^{\infty} \frac{1}{n^2-7n-8}$

<u>soln:</u>

Note that $n^2-7n-8=(n-8)(n+1)>0 \quad \forall n\geq 10$.

Say $a_n = \frac{1}{n^2-7n-8} \quad \forall n\geq 10$. So, $a_n>0 \quad \forall n\geq 10$.

<u>Recall</u> (The Limit comparison test)

Suppose that $\sum_{n=1}^{\infty} a_n$ and $\sum_{n=1}^{\infty} b_n$ are series with positive terms

(i) If $\lim_{n\to\infty} \frac{a_n}{b_n} = L$ and $L\neq 0$ & $L\neq\infty$, then $\sum_{n=1}^{\infty} a_n$ and $\sum_{n=1}^{\infty} b_n$ both converge and both divege

(ii) If $\lim_{n\to\infty} \frac{a_n}{b_n} = 0$ and $\sum_{n=1}^{\infty} b_n$ is convergent, then $\sum_{n=1}^{\infty} a_n$ is also convergent.

(iii) If $\lim_{n\to\infty} \frac{a_n}{b_n} = \infty$ and $\sum_{n=1}^{\infty} b_n$ is divergent, then $\sum_{n=1}^{\infty} a_n$ is also divergent

Now, consider $\sum_{n=10}^{\infty} \frac{1}{n^2}$ where $b_n = \frac{1}{n^2} > 0 \quad \forall n \geq 10$

Then, $\lim_{n\to\infty} \frac{a_n}{b_n} = \lim_{n\to\infty} \frac{\frac{1}{n^2-7n-8}}{\frac{1}{n^2}}$

$$= \lim_{n\to\infty} \frac{n^2}{n^2-7n-8}$$

$$= \lim_{n\to\infty} \frac{\cancel{n^2}}{\cancel{n^2}(1-\underbrace{\tfrac{7}{n}}_{0}-\underbrace{\tfrac{8}{n^2}}_{0})} = 1 < \infty.$$

and $\sum_{n=10}^{\infty} \frac{1}{n^2}$ is convergent by p-series test.
$(p=2>1)$.

Then, $\sum_{n=10}^{\infty} a_n$ is convergent by LCT.

Recall: (p-series test)

$$\sum_{n=1}^{\infty} \frac{1}{n^p} = \begin{cases} \text{conv.} & \text{if } p>1 \\ \text{div. to } \infty & \text{if } p \leq 1 \end{cases}$$

f) $\sum_{n=0}^{\infty} \frac{\sqrt[4]{2n^3+n}}{\sqrt[3]{1+3n^5}}$

Soln:

Say $a_n = \frac{\sqrt[4]{2n^3+n}}{\sqrt[3]{1+3n^5}} > 0 \quad \forall n \geq 0$

Consider $\sum_{n=1}^{\infty} \frac{1}{n^{11/12}}$ where $b_n = \frac{1}{n^{11/12}} \quad \forall n \geq 1.$

Then, $\lim_{n\to\infty} \frac{a_n}{b_n} = \lim_{n\to\infty} \frac{\sqrt[4]{n^3(2+\frac{1}{n^2})}}{\sqrt[3]{n^5(1/n^5+3)}} \cdot \frac{n^{11/12}}{1}$

$$= \lim_{n\to\infty} \frac{\cancel{n^{3/4}} \cdot \sqrt[4]{2+1/n^2} \cdot \cancel{n^{11/12}}}{\cancel{n^{5/3}} \cdot \sqrt[3]{1/n^5+3}} = \frac{2^{1/4}}{3^{1/3}} \neq 0$$

$\sum_{n=1}^{\infty} \frac{1}{n^{11/12}}$ is divergent by p-series test

$(p = 11/12 < 1)$

So, $\sum_{n=1}^{\infty} \frac{\sqrt[4]{2n^3+n}}{\sqrt[3]{1+3n^5}}$ is also divergent by LCT.

$\Rightarrow \sum_{n=0}^{\infty} \frac{\sqrt[4]{2n^3+n}}{\sqrt[3]{1+3n^5}}$ is divergent.