MATH 120 – 2021-2 – RECITATION PROBLEMS – W13

1) For a twice differentiable vector valued function $r(t) = (x(t), y(t), z(t))$ if $|r(t)|$ is constant, show that the vectors $r(t)$ and $r'(t)$ are orthogonal.

Soln:

Suppose $|r(t)|$ is constant.

i.e. $|r(t)| = \sqrt{r(t) \cdot r(t)} = c$

$$|r(t)|^2 = r(t) \cdot r(t) = c^2$$

$$\Rightarrow r'(t) \cdot r(t) + r(t) \cdot r'(t) = 0$$

$$\Rightarrow 2r(t) \cdot r'(t) = 0$$

$$\Rightarrow r(t) \cdot r'(t) = 0$$

So, $r(t)$ and $r'(t)$ are orthogonal.

(2) For diff'ble vector valued function $u(t)$ and $v(t)$. Suppose that $u'(t) = au(t) + bv(t)$, $v'(t) = cu(t) - av(t)$ for some constant a, b, c.

Show that the vector $u(t) \times v(t)$ is constant.

Soln:

$$\frac{d}{dt}\left(u(t) \times v(t)\right) = u'(t) \times v(t) + u(t) \times v'(t)$$

$$= \left(au(t) + bv(t)\right) \times v(t) + u(t) \times \left(cu(t) - av(t)\right)$$

$$= \cancel{a u(t) \times v(t)} + b\, v(t) \times v(t) + c u(t) \times u(t) - \cancel{a u(t) \times v(t)}$$

$$= c u(t) \times u(t) + b v(t) \times v(t)$$

$$= c \cdot \vec{0} + b \cdot \vec{0} = \vec{0}$$

So, $u(t) \times v(t)$ is constant.

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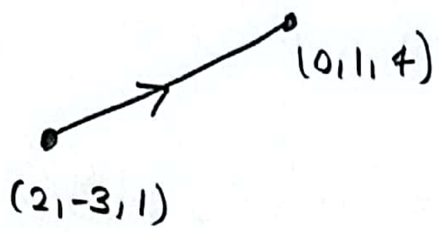
a) Parametrize the line segment from the point $(2,-3,1)$ to the point $(0,1,4)$.

Soln:



$V = (0,1,4) - (2,-3,1)$

$= \langle -2, 4, 3 \rangle$

$\ell(t) = \langle 2,-3,1 \rangle + t \cdot \langle -2,4,3 \rangle \ , t \in [0,1]$

$\ell(t) = \langle \underbrace{2-2t}_{x(t)}, \underbrace{-3+4t}_{y(t)}, \underbrace{1+3t}_{z(t)} \rangle$

$\ell(0) = (2,-3,1)$

$\ell(1) = (0,1,4)$

b) Parametrize the piece of the circle $x^2+y^2=2y$ from the point $(1,1)$ to the point $(0,2)$ in counterclockw direction.

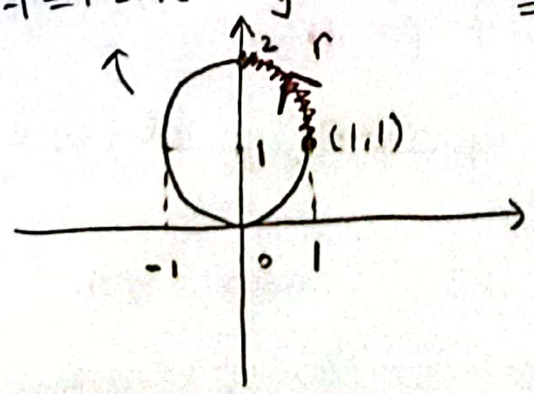
Soln: $x^2+y^2=2y \Rightarrow x^2+y^2-2y=0$

$\Rightarrow x^2+(y-1)^2-1=0$

$\Rightarrow x^2+(y-1)^2=1$

$x = r \cdot \cos\theta \quad r=1$

$y-1 = r\sin\theta$



Circle with center $(0,1)$.

$r(\theta) = \langle \cos\theta, \sin\theta + 1 \rangle$

where $\theta \in [0, \pi/2]$

$r(0) = (1,1)$ & $r(\frac{\pi}{2}) = (0,2)$

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(C) Parametrize the curve of intersection of the ellipsoid $z = 16 - x^2 - y^2$ and the plane $z = 16 - 2y$ in the direction of your choosing.

Soln:

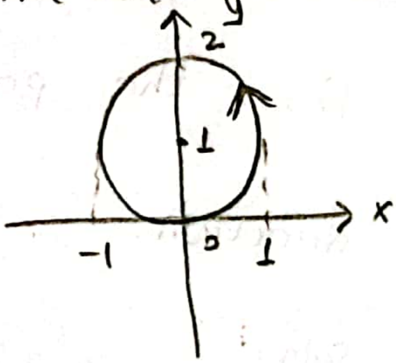
$$z = 16 - x^2 - y^2 \quad \& \quad z = 16 - 2y$$

$$\Rightarrow \cancel{16} - 2y = \cancel{16} - x^2 - y^2$$

$$\Rightarrow x^2 + y^2 - 2y = 0$$

$$\Rightarrow x^2 + (y-1)^2 - 1 = 0$$

$$\Rightarrow \underbrace{x^2 + (y-1)^2 = 1}_{C:}$$ (circle with center (0,1).



$x = \cos\theta$
$y = 1 + \sin\theta$
$\theta \in [0, 2\pi]$

$$\Rightarrow z = 16 - 2y$$
$$= 16 - 2.(1 + \sin\theta)$$
$$= 16 - 2 - 2\sin\theta$$
$$= 14 - 2\sin\theta$$

$\therefore$ $C(\theta) = \langle \cos\theta, 1 + \sin\theta, 14 - 2\sin\theta \rangle$, $\theta \in [0, 2\pi]$

④ Find a potential function $\phi(x,y)$ to show that the plane vector field $\vec{F}(x,y) = \langle 2x + ye^{xy}, 2y + xe^{xy} \rangle$ is conservative.

Soln

A vector fied F is conservative in D if

$\vec{F}(x,y,z) = \vec{\nabla}\phi(x,y,z)$ in D for some ϕ, called potential function.
(or (x,y) ; (x,y))

Let ϕ be a potential function of F, i.e. $\vec{F}(x,y) = \vec{\nabla}\phi(x,y)$

$$\frac{\partial \phi}{\partial x} = 2x + ye^{xy} \quad \text{and} \quad \frac{\partial \phi}{\partial y} = 2y + xe^{xy}$$

$$\phi(x,y) = \int \frac{\partial \phi}{\partial x}\,dx = \int (2x + ye^{xy})\,dx = x^2 + e^{xy} + f(y)$$

$$\frac{\partial \phi}{\partial y} = \frac{\partial (x^2 + e^{xy} + f(y))}{\partial y} = xe^{xy} + f'(y) = 2y + xe^{xy}$$

$\Rightarrow f'(y) = 2y \Rightarrow f(y) = y^2 + c$ for some $c \in \mathbb{R}$

Then, $\phi(x,y) = x^2 + e^{xy} + y^2 + c$, $c \in \mathbb{R}$

Take $c = 0$. Then, $\phi(x,y) = x^2 + e^{xy} + y^2$ is a potential function, So, $\vec{F}$ is conservative.

(5) Show that the vector valued

$$\vec{F}(x,y) = \langle xy^2 + x^3y,\ (x+y)x^2 \rangle$$

is not conservative.

Soln:

Necessary condition for a Conservative plane:

If $F(x,y) = F_1(x,y)\,\vec{i} + F_2(x,y)\,j$ is conservative in D

then the condition $\frac{\partial F_1}{\partial y} = \frac{\partial F_2}{\partial x}$ must be satisfied

at all pts in D. Converse is not true always!

Say $F_1(x,y) = xy^2 + x^3y$ and $\bar{F}_2(x,y) = x^3 + x^2y$

$$\frac{\partial F_1}{\partial y} = 2xy + x^3 \neq 3x^2 + 2xy = \frac{\partial F_2}{\partial y} \text{ on } \mathbb{R}^2.$$

So, by the necessary condition for a conservative plane,

F is not Conservative.

(6) Find a potential function $\phi(x,y,z)$ so that the space vector field

$$\vec{F}(x,y,z) = \left\langle \frac{y}{1+x^2} + 2xyz,\ \arctan x + x^2z + ze^{yz},\ x^2y + ye^{yz} \right\rangle$$

is conservative.

Soln: $\vec{F}(x,y,z) = \nabla\phi(x,y,z)$

$$\frac{\partial \phi}{\partial x} = \frac{y}{1+x^2} + 2xyz, \quad \frac{\partial \phi}{\partial y} = \arctan x + x^2z + ze^{yz} \text{ and}$$

$$\frac{\partial \phi}{\partial z} = x^2y + ye^{yz}.$$

$$\phi(x,y,z) = \int \frac{\partial \phi}{\partial x}\, dx = \int \left(\frac{y}{1+x^2} + 2xyz\right) dx$$

$$= y\arctan x + x^2yz + f(y,z)$$

$$\frac{\partial \phi}{\partial y} = \frac{\partial \left(y\arctan x + x^2yz + f(y,z)\right)}{\partial y}$$

$$= \arctan x + x^2z + f_y(y,z) = \arctan x + x^2z + ze^{yz}$$

$$\Rightarrow f_y(y,z) = ze^{yz}$$

$$\Rightarrow f(y,z) = e^{yz} + g(z).$$

$$\frac{\partial \phi}{\partial z} = \frac{\partial (y \arctan x + x^2yz + e^{yz} + g(z))}{\partial z}$$

$$= x^2y + ye^{yz} + g'(z) = x^2y + ye^{yz},$$

$$\Rightarrow \quad g'(z) = 0 \quad \Rightarrow \quad g(z) = c \text{ for some } c \in \mathbb{R}.$$

Take $c=0$.

So, $\phi(x,y,z) = y\arctan x + x^2yz + e^{yz}$ is a potential function for $\vec{F}$.

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