Math 120 Recitation Problems Week 11

1) Find the absolute maximum and minimum values of the function $\boxed{f(x,y,z) = 2x+3y+z}$ subject to the unit sphere $x^2+y^2+z^2=1$.

$g(x,y,z) = x^2+y^2+z^2-1$
constraint function

Solution:

[By the Theorem 4 of Chapter 13, we need to find the critical points of the Lagrange function

$L(x,y,z,\lambda) = f(x,y,z) + \lambda\, g(x,y,z)$.]

By the method of Lagrange multipliers, we need to find the critical points P of L, that is,

$$\nabla L \big|_P = \vec{0} \iff \nabla f = \lambda \nabla g, \quad \lambda \neq 0.$$

$$\boxed{L = 2x+3y+z + \lambda(x^2+y^2+z^2-1)}$$

$$\frac{\partial L}{\partial x} = 0 \iff 2+2\lambda x = 0 \iff x = -\frac{1}{\lambda}$$

$$\frac{\partial L}{\partial y} = 0 \iff 3+2\lambda y = 0 \iff y = -\frac{3}{2\lambda}$$

$$\frac{\partial L}{\partial z} = 0 \iff 1+2\lambda z = 0 \iff z = -\frac{1}{2\lambda}$$

$(*)$, $\lambda \neq 0$

$$\frac{\partial L}{\partial \lambda} = 0 \iff x^2+y^2+z^2-1=0 \quad \text{(constraint eqn.)}$$

Put the equations in $(*)$ onto the constraint eqn so that

$$\frac{1}{\lambda^2} + \frac{9}{4\lambda^2} + \frac{1}{4\lambda^2} - 1 = 0 \iff \lambda = \mp \frac{\sqrt{14}}{2}$$

For $\lambda = \frac{\sqrt{14}}{2}$, we have $P = (x,y,z) = \left(-\frac{2}{\sqrt{14}}, \frac{-3}{\sqrt{14}}, \frac{-1}{\sqrt{14}}\right)$

$$\Rightarrow f(x,y,z)\Big|_P = \frac{-4-9-1}{\sqrt{14}} = \boxed{-\sqrt{14}. \Rightarrow \text{abs min.}}$$

For $\lambda = \frac{-\sqrt{14}}{2}$, we have $P = (x,y,z) = \left(\frac{2}{\sqrt{14}}, \frac{3}{\sqrt{14}}, \frac{1}{\sqrt{14}}\right)$

$$\Rightarrow f(x,y,z)\Big|_P = \boxed{\sqrt{14} \Rightarrow \text{abs max.}}$$

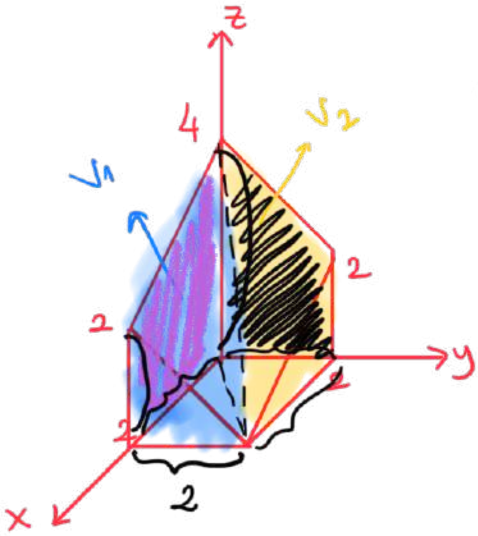
2 Let D be the region given by $0 \le x \le 2$ and $0 \le y \le 2$. Evaluate $\boxed{I = \iint_D (4-x-y)\,dA}$ by interpreting it as a volume. Seperate D into four unit squares in the obvious way. Calculate the Riemann sum for each of the following set of sample points. Compare these values with I.

(a) the <u>lower left corner of the squares,</u>

(b) the centers of unit squares, ex

(c) the upper right corners of unit squares. ex

Solution:



I represents a volume of the solid lying vertically above D and below the surface $z = 4 - x - y$.

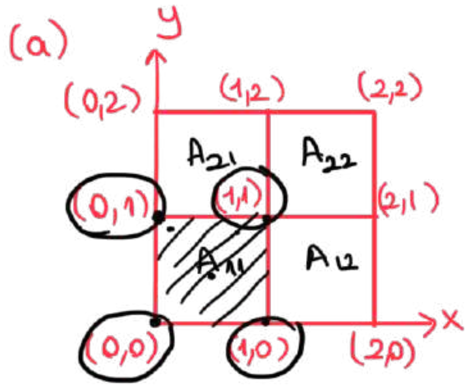
$$I = V_1 + V_2$$

= sum of the volumes of two pyramids

$$= \underbrace{\frac{4+2}{2} \cdot 2}_{\text{base area}} \cdot \underbrace{2}_{\text{height}} \cdot \frac{1}{3} + \frac{4+2}{2} \cdot 2 \cdot 2 \cdot \frac{1}{3}$$

$$= \boxed{8.}$$

(a)



$\Delta A_{ij} \rightarrow$ area of subrectangle

$\Delta A_{ij} = 1$ (since we have unit squares)

$$R(f, P) = \sum_{i=1}^{m} \sum_{j=1}^{n} f(x_{ij}^*, y_{ij}^*) \Delta A_{ij}$$

$$= f(0,0) \cdot 1 + f(1,0) \cdot 1 + f(0,1) \cdot 1 + f(1,1) \cdot 1$$

$$= (4-0-0) \cdot 1 + (4-1-0) \cdot 1 + (4-0-1) \cdot 1 + (4-1-1) \cdot 1$$

$$= \boxed{12}.$$

③ For each of the following double integrals, sketch the domain of integration, express the integral in both dxdy and dydx iterations and evaluate the integral.

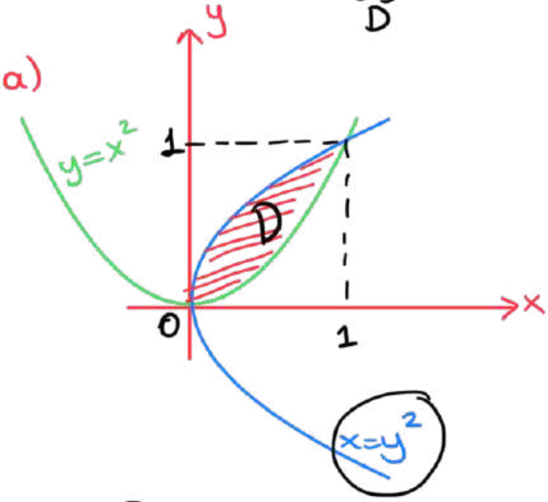
(a) $\iint_D 1\, dA$ where D is the finite region between $y=x^2$ and $x=y^2$.

(b) $\int_0^1 \int_{2y}^{2} \cos(x^2)\, dx\, dy$

(c) $\iint_R xy\, dA$ where R is the finite region between $xy=1$ and $3x+y=4$.

Solution: $I=\iint_D 1\, dA$ = Area of the region D.

(a)

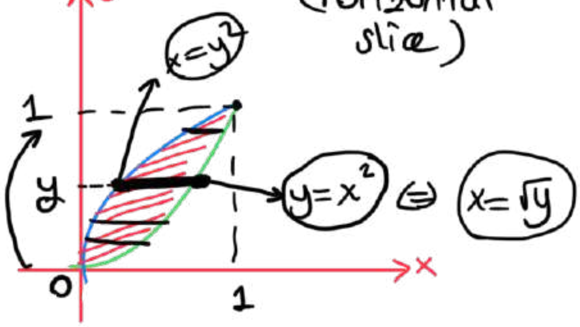


Intersection points:

$$\left.\begin{array}{l} y=x^2 \\ x=y^2 \end{array}\right\} \Rightarrow \begin{array}{l} y-y^4=0 \\ y=0 \text{ or } y=1. \end{array}$$

$\Rightarrow$ $dx\,dy$ order:

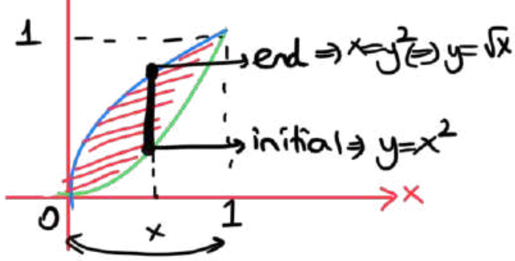
(horizontal slice)



$$I=\int_0^1 \int_{x=y^2}^{\sqrt{y}} 1\, dx\, dy$$

$dy\,dx$ order:

(vertical slice)



$$I=\int_0^1 \int_{x^2}^{\sqrt{x}} 1 \cdot dy\, dx$$

$$= \int_0^1 \left(x \Big|_{x=y^2}^{x=\sqrt{y}} \right) dy \qquad\qquad = \int_0^1 \left(y \Big|_{y=x^2}^{y=\sqrt{x}} \right) dx$$

$$= \int_0^1 (\sqrt{y} - y^2)\, dy \qquad\qquad = \int_0^1 (\sqrt{x} - x^2)\, dx$$

ex:

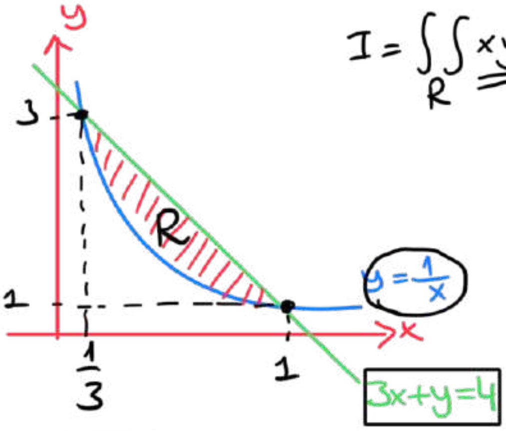
$$= \frac{1}{3} \qquad\qquad = \frac{1}{3}$$

(b) Exercise (Answer: $I = \frac{\sin 4}{4}$)

$$I = \int_0^1 \int_{2y}^2 \cos(x^2)\, dx\, dy$$

Note that we cannot find an antiderivative of $\cos(x^2)$, that's why we have to convert $dx\,dy$ into $dy\,dx$ order to be able to evaluate I.

(c)



$$I = \iint_R xy \, dA = ?$$

Intersection:

$$\left.\begin{array}{l} y = \frac{1}{x} \\ 3x + y = 4 \end{array}\right\} \quad 3x + \frac{1}{x} = 4 \iff \frac{3x^2+1}{x} - 4 = 0$$

$$\iff (3x-1)(x-1) = 0$$

$$\iff x = \frac{1}{3} \text{ or } x = 1$$
$$\quad y = 3 \qquad\quad y = 1$$

dxdy order:

(horizontal slice)

$3x+y=4 \iff x = \frac{4-y}{3}$

dydx order:

(vertical slice)

dxdy order:

$$I = \int_{1}^{3} \int_{\frac{1}{y}}^{4-\frac{y}{3}} xy \, dx \, dy$$

ex

$$= \frac{22}{27} - \frac{\ln 3}{2}.$$

dydx order:

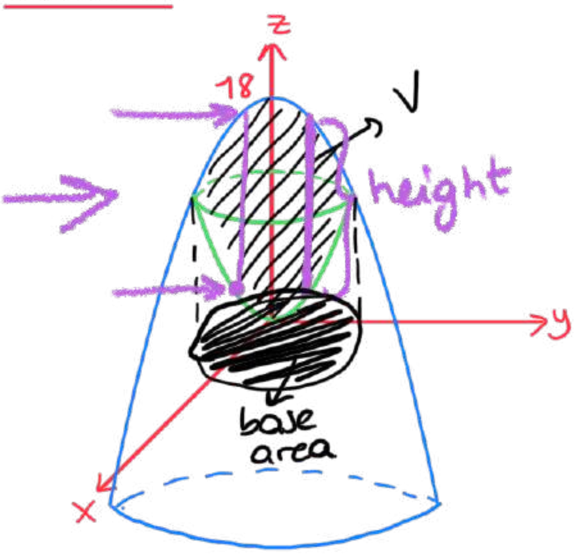
$$I = \int_{\frac{1}{3}}^{1} \int_{\frac{1}{x}}^{4-3x} xy \, dy \, dx$$

ex

$$= \frac{22}{27} - \frac{\ln 3}{2}.$$

4) Express the volume of the solid lying between the paraboloids $z = x^2+y^2$ and $z = 18-x^2-y^2$.

Solution:



Intersection:

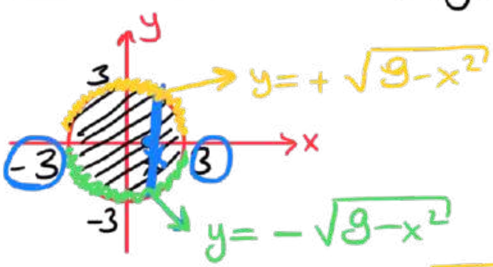
$$\left.\begin{array}{l} z = 18-x^2-y^2 \\ z = x^2+y^2 \end{array}\right\} \quad 18-x^2-y^2 = x^2+y^2$$

base area

$$\Rightarrow 2x^2+2y^2 = 18 \quad \Rightarrow \boxed{x^2+y^2=9}$$

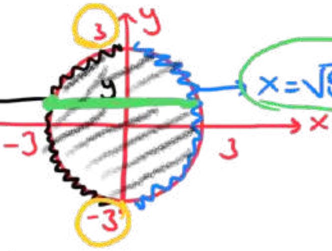
we have a circle centered at origin with radius 3.

Base area: $x^2+y^2=9 \Rightarrow y = \mp\sqrt{9-x^2}$



$$\text{Volume} = V = \int_{-3}^{3} \int_{-\sqrt{9-x^2}}^{\sqrt{9-x^2}} \left[18-x^2-y^2-(x^2+y^2)\right] dy\,dx$$

2nd way:



$x^2+y^2=9 \Rightarrow x=\mp\sqrt{9-y^2}$

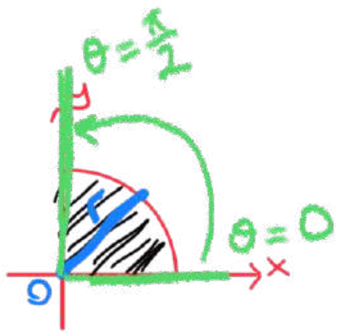
$$V=\int_{-3}^{3}\int_{-\sqrt{9-y^2}}^{\sqrt{9-y^2}}\left[18-x^2-y^2-(x^2+y^2)\right]dx\,dy$$

⑤ Express the following integrals in polar coordinates.

(a) $\displaystyle\int_0^1\int_0^{\sqrt{1-y^2}}\underbrace{(x^2+y^2)}\,dx\,dy$ (b) $\displaystyle\int_{-\frac{\sqrt3}{2}}^{\frac{\sqrt3}{2}}\int_{\frac12}^{\sqrt{1-x^2}}\frac{x+y}{x}\,dy\,dx$

Solution:

(a) $x=r\cos\theta$
$y=r\sin\theta$
$x^2+y^2=r^2$
and $\boxed{\overset{dA}{dx\,dy}=r\,dr\,d\theta}$



$0\le x\le\sqrt{1-y^2} \Leftrightarrow \underbrace{x^2+y^2=1}$.
$\Updownarrow$
$r^2=1$
$r=1$

$0\le y\le 1$

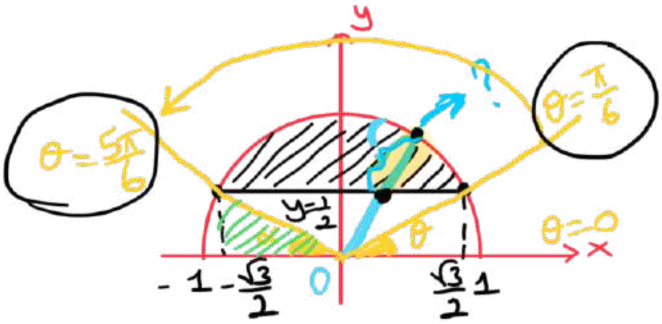
$$I=\int_0^1\int_0^{\sqrt{1-y^2}}\underbrace{(x^2+y^2)}_{=r^2}\,dx\,dy=\int_0^{\frac{\pi}{2}}\int_0^1 r^2\,r\,dr\,d\theta$$

(b) $\dfrac{x+y}{x}=1+\dfrac{y}{x}=1+\tan\theta \rightarrow$ integrand in polar coordinate ✓

$\frac12\le y\le\sqrt{1-x^2} \Rightarrow y=\sqrt{1-x^2} \Rightarrow x^2+y^2=1 \Leftrightarrow r^2=1 \Leftrightarrow \boxed{r=1}$

$-\frac{\sqrt3}{2}\le x\le\frac{\sqrt3}{2} \Rightarrow$ when $y=\frac12 \Rightarrow x^2+\frac14=1 \Rightarrow x^2=\frac34$

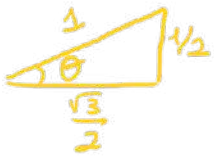
$\Rightarrow x = \mp \frac{\sqrt{3}}{2}.$



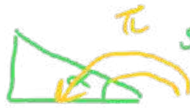
At $y=\frac{1}{2} \Leftrightarrow \frac{1}{2} = r\sin\theta$

$r = \frac{1}{2\sin\theta}$ Initial

$$I = \int_{-\frac{\sqrt{3}}{2}}^{\frac{\sqrt{3}}{2}} \int_{\frac{1}{2}}^{\sqrt{1-x^2}} \frac{x+y}{x}\, dy\, dx = \int_{\frac{\pi}{6}}^{\frac{5\pi}{6}} \int_{\frac{1}{2\sin\theta}}^{1} (1+\tan\theta)\, r\, dr\, d\theta$$



$\sin\theta = \frac{1}{2} \Leftrightarrow \theta = \frac{\pi}{6}.$



$\sin\alpha = \frac{1}{2}$

$\alpha = \frac{\pi}{6}$

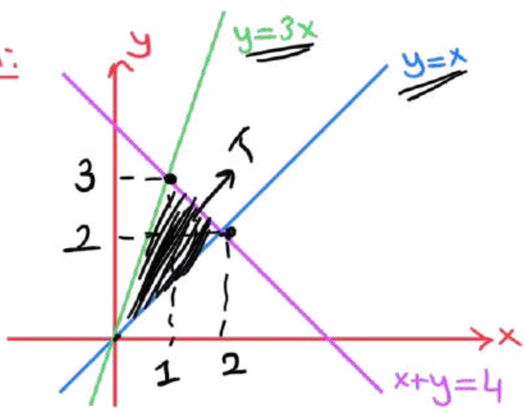
$\pi - \alpha = \pi - \frac{\pi}{6} = \frac{5\pi}{6}$

(6) Let T be the triangular region bounded by the lines $y=x$, $y=3x$, and $x+y=4$. Consider the integral

$$I = \iint_T (x+y)\, dA.$$

Sketch the region T and express I by using the iterations $dx\,dy$ and $dy\,dx$. Compute I using the substitutions $x = u - v$ and $y = u + v$. Explain why these substitutions simplifies the computation of I.

Solution:

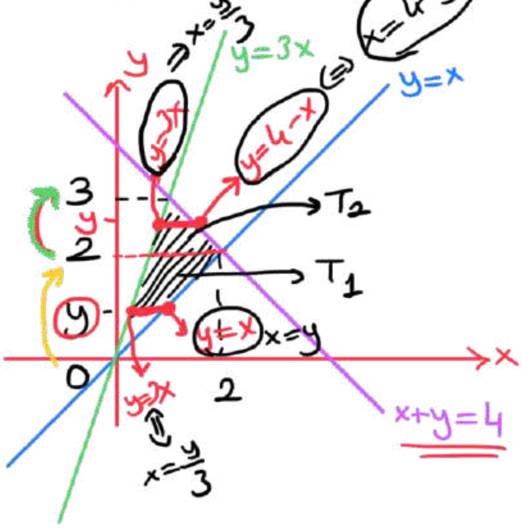


Intersections:

$\left.\begin{array}{l} y=x \\ x+y=4 \end{array}\right\}$ $2x=4$, $x=2$, $y=2$

$\left.\begin{array}{l} y=3x \\ x+y=4 \end{array}\right\}$ $4x=4$, $x=1$, $y=3$

dxdy order:



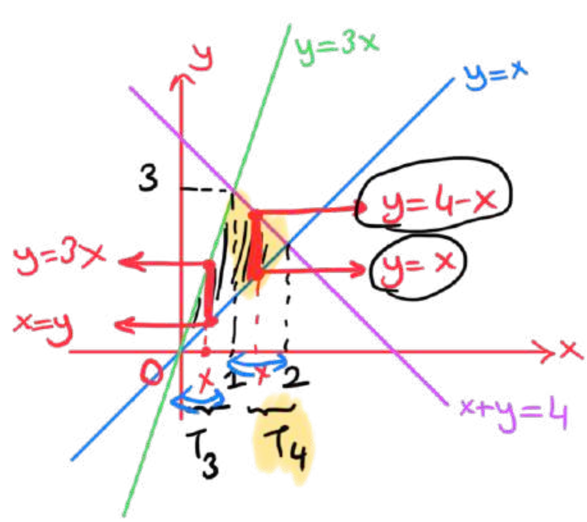
$$I=\iint_{T_1}(x+y)\,dx\,dy+\iint_{T_2}(x+y)\,dx\,dy$$

$$=\underbrace{\int_0^2\int_{\frac{y}{3}}^{y}(x+y)\,dx\,dy}+\underbrace{\int_2^3\int_{\frac{y}{3}}^{4-y}(x+y)\,dx\,dy}$$

⋮ ex.

$$=\frac{16}{3}.$$

dydx order:



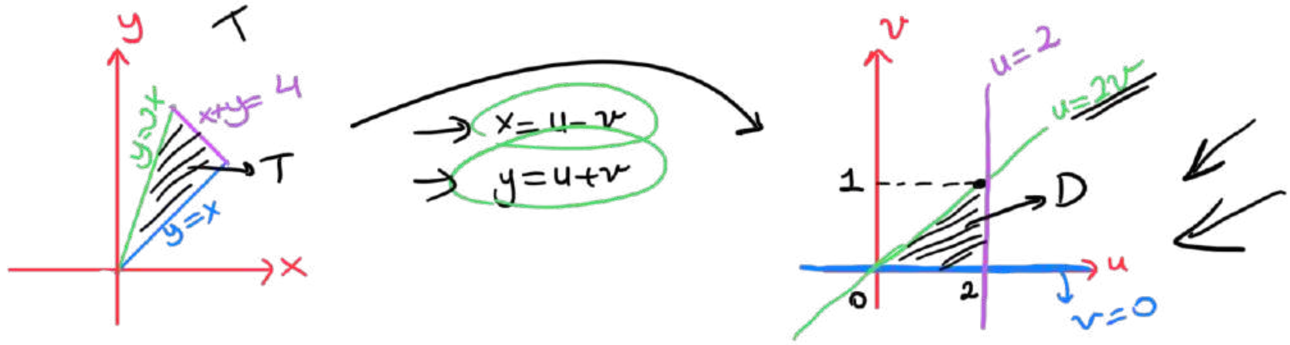
$$I=\iint_{T_3}(x+y)\,dy\,dx+\iint_{T_4}(x+y)\,dy\,dx$$

$$=\int_0^1\int_x^{3x}(x+y)\,dy\,dx+\int_1^2\int_x^{4-x}(x+y)\,dy\,dx$$

⋮ ex

$$=\frac{16}{3}.$$

Let's use the substitutions $x=u-v$ and $y=u+v$.



$$y=x \Rightarrow \cancel{u}+v=\cancel{u}-v \Rightarrow 2v=0 \Rightarrow v=0$$

$y = 3x \Rightarrow u+v = 3(u-v) \Rightarrow u+v = 3u-3v \Rightarrow \boxed{u=2v}$

$x+y=4 \Rightarrow u-\cancel{v}+u+\cancel{v} = 4 \Rightarrow 2u=4 \Rightarrow u=2$

So,

$$I = \iint_T (\underset{u-v}{x}+\underset{u+v}{y})\,dA = \iint_D (u-v+u+v)\left|\frac{\partial(x,y)}{\partial(u,v)}\right| du\,dv$$

since $dA = \left|\frac{\partial(x,y)}{\partial(u,v)}\right| du\,dv$. That is,

$$\boxed{I = \iint_D 2u \cdot \left|\frac{\partial(x,y)}{\partial(u,v)}\right| du\,dv}$$

$$\left|\frac{\partial(x,y)}{\partial(u,v)}\right| = \left|\,\begin{vmatrix} \frac{\partial x}{\partial u} & \frac{\partial x}{\partial v} \\ \frac{\partial y}{\partial u} & \frac{\partial y}{\partial v} \end{vmatrix}\,\right| = \left|\,\underbrace{\begin{vmatrix} 1 & -1 \\ 1 & 1 \end{vmatrix}}_{1\cdot 1-(-1)\cdot 1 = 2}\,\right| = |2| = 2.$$

abs. value

determinant

$x = u-v \Rightarrow \frac{\partial x}{\partial u} = 1$ and $\frac{\partial x}{\partial v} = -1$

$y = u+v \Rightarrow \frac{\partial y}{\partial u} = 1$ and $\frac{\partial y}{\partial v} = 1$

Hence,

$$I = \int_0^1 \int_{2v}^{2} 2u \cdot 2\, du\,dv$$

ex $= \frac{16}{3}$