MATH 120 - 2021-2 RECITATION PROBLEMS - WEEK 13

① Find the directional derivative of the function $f(x,y,z) = x^2yz + yz^2$ at the point $(1,1,1)$ in the direction normal to the surface $x^2 - xyz + z^2 - 1 = 0$.

Soln:

A normal vector to the surface $x^2 - xyz + z^2 - 1 = 0$ is $\vec{n} = \vec{\nabla} g(1,1,1) = \left\langle \frac{\partial g}{\partial x}\Big|_{(1,1,1)}, \frac{\partial g}{\partial y}\Big|_{(1,1,1)}, \frac{\partial g}{\partial z}\Big|_{(1,1,1)} \right\rangle$

where $g(x,y,z) = x^2 - xyz + z^2 - 1$

i.e. $\vec{n} = \left\langle 2x - yz\Big|_{(1,1,1)}, -xz\Big|_{(1,1,1)}, -xy + 2z\Big|_{(1,1,1)} \right\rangle$

$= \langle 1, -1, 1 \rangle.$

The directional derivative of the function $f(x,y,z) = x^2yz + yz^2$ at $(1,1,1)$ in the direction $\vec{n} = \langle 1,-1,1 \rangle$ is $D_{\vec{n}} f(1,1) = \vec{\nabla} f(1,1) \cdot \frac{\vec{n}}{\|\vec{n}\|}$

since f is differentiable function everywhere

$\Rightarrow$ (f, f_x and f_y are polynomials which are cont. everywhere)

$$D_{\vec{n}} f(1,1,1) = \vec{\nabla} f(1,1,1) \cdot \frac{\vec{n}}{\|\vec{n}\|}$$

$$= \left\langle 2xyz \Big|_{(1,1,1)}, \; x^2z + z^2 \Big|_{(1,1,1)}, \; x^2y + 2yz \Big|_{(1,1,1)} \right\rangle \cdot \frac{\langle 1, -1, 1 \rangle}{\sqrt{3}}$$

$$= \langle 2, 2, 3 \rangle \cdot \left\langle \frac{1}{\sqrt{3}}, \frac{-1}{\sqrt{3}}, \frac{1}{\sqrt{3}} \right\rangle$$

$$= \frac{2}{\sqrt{3}} - \frac{2}{\sqrt{3}} + \frac{3}{\sqrt{3}} = \frac{3}{\sqrt{3}} = \sqrt{3}$$

! "The Fundamental Lemma"

Suppose, f_x, f_y are cont. at (x_0, y_0). Then,

f is diff'ble at (x_0, y_0).

<u>Remark</u>: If f_x or f_y are not cont at (x_0, y_0) then

we <u>cannot</u> say f is not diff'ble at (x_0, y_0).

(2) Find the directional derivative of the function $f(x,y,z) = \dfrac{y}{x+z}$ at $(1,1,0)$ in the direction for which the function $g(x,y,z) = \ln(x^2+y^2+z^2)$ increases most rapidly at the same point.

Soln:

Note that if f is diff'ble, then the maximum value of the directional derivative $D_{\vec{u}} f(x)$ is $|\vec{\nabla} f(x)|$.

$$D_{\vec{u}} f(x) = \vec{\nabla} f(x) \cdot \vec{u} = |\vec{\nabla} f(x)| \cdot \underset{\underset{1}{\parallel}}{|u|} \cdot \overbrace{\cos\alpha}^{\max = 1.\ \Rightarrow\ \alpha = 0}$$

(since $\vec{u}$ is unit vector)

where α is the angle between $\vec{\nabla} f$ and $\vec{u}$

This occurs if $\vec{u}$ and $\vec{\nabla} f(x)$ have the same direction

The direction for which g increases most rapidly at $(1,1,0)$ is $\vec{u} = \vec{\nabla} g(1,1,0)$.

i.e. $\vec{u} = \vec{\nabla} g(1,1,0) = \left\langle \left.\dfrac{\partial g}{\partial x}\right|_{(1,1,0)}, \left.\dfrac{\partial g}{\partial y}\right|_{(1,1,0)}, \left.\dfrac{\partial g}{\partial z}\right|_{(1,1,0)} \right\rangle = \langle 1,1,0\rangle$

$\ldots\ 2y\ ,\ 2z\ ,\ \rangle = \langle 1,1,0\rangle$

Since rational functions are continuous on their domains,

f_x, f_y, f_z are cont. on their domains. So, f is diff'ble.

$$D_{\vec{u}} f(1,1,0) = \vec{\nabla f}(1,1,0) \cdot \frac{\langle 1,1,0\rangle}{\sqrt{2}}$$

$$= \left\langle \left.\frac{\partial f}{\partial x}\right|_{(1,1,0)}, \left.\frac{\partial f}{\partial y}\right|_{(1,1,0)}, \left.\frac{\partial f}{\partial z}\right|_{(1,1,0)} \right\rangle \cdot \left\langle \frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}, 0 \right\rangle$$

$$= \left\langle \left.-\frac{y}{(x+z)^2}\right|_{(1,1,0)}, \left.\frac{1}{x+z}\right|_{(1,1,0)}, \left.\frac{-y}{(x+z)^2}\right|_{(1,1,0)} \right\rangle \cdot \left\langle \frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}, 0 \right\rangle$$

$$= \boxed{\langle -1, 1, -1 \rangle \cdot \left\langle \frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}, 0 \right\rangle}$$

$\left\langle \frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}, 0 \right\rangle$

$$= \cancel{-\frac{1}{\sqrt{3}} + \frac{1}{\sqrt{3}} + 0 = 0}$$

$\|$

$$-\frac{1}{\sqrt{2}} + \frac{1}{\sqrt{2}} + 0 = 0$$

So, $\vec{\nabla f}(1,1,0) \perp \vec{\nabla g}(1,1,0)$

③ Consider the function $f(x,y) = xe^{xy}$ at the point $(\sqrt{2},0)$. Find all direction vectors u so that $D_{\vec{u}} f(\sqrt{2},0) = 2$.

soln:

The function is the composition of polynomials and exponential function, so f, f_x, f_y are cont. everywhere

So, f is diff'ble everywhere.

$2 = D_{\vec{u}} f(\sqrt{2},0) = \vec{\nabla} f(\sqrt{2},0) \cdot \vec{u}$ where $\vec{u} = \langle a, b \rangle$

and $|\vec{u}| = \sqrt{a^2+b^2} = 1$

Then,

$$2 = \left\langle e^{xy} + xye^{xy}\Big|_{(\sqrt{2},0)},\ x^2 e^{xy}\Big|_{(\sqrt{2},0)} \right\rangle \cdot \langle a, b \rangle$$

$$= \langle 1, 2 \rangle \cdot \langle a, b \rangle$$

$$= a + 2b,$$

So, we have $\underset{①}{a+2b=2}$ and $\underset{②}{a^2+b^2=1}$.

① $\Rightarrow a = 2-2b$ ② $\Rightarrow 4(1-b)^2 + b^2 = 1 \Rightarrow 4(1-2b+b^2) + b^2 = 1$

$\Rightarrow 4 - 8b + 5b^2 = 1$

$\Rightarrow 5b^2 - 8b + 3 = 0$

$\Rightarrow \boxed{(5b-3)(b-1) = 0}$ $b = \frac{3}{5}$ OR $b = 1$

$b = 3/5 \Rightarrow a = 2 - \frac{6}{5} = \frac{4}{5} \Rightarrow \vec{u} = \langle 4/5, 3/5 \rangle$

④ Compute the directional derivative of the function

$f(x,y) = \dfrac{1}{1+x^2+5y^2}$ at the point $(2,1)$ in the direction

$u = 5\vec{i}+12\vec{j}$. Determine the direction at the point $(3,0)$

in which the rate of change of f is the largest

and compute this rate of change.

Soln:

Since f is a rational function, f, f_x, f_y are cont. on their domains, so f is diff'ble on Dom f

$$D_{\vec{u}} f(2,1) = \vec{\nabla} f(2,1) \cdot \frac{\langle 5,12 \rangle}{13}$$

$$= \left\langle \left.\frac{\partial f}{\partial x}\right|_{(2,1)}, \left.\frac{\partial f}{\partial y}\right|_{(2,1)} \right\rangle \cdot \left\langle \frac{5}{13}, \frac{12}{13} \right\rangle$$

$$= \left\langle \left.\frac{-2x}{(1+x^2+5y^2)^2}\right|_{(2,1)}, \left.\frac{-10y}{(1+x^2+5y^2)^2}\right|_{(2,1)} \right\rangle \cdot \left\langle \frac{5}{13}, \frac{12}{13} \right\rangle$$

$1+4+5$

$$= \left\langle \frac{-4}{100}, \frac{-10}{100} \right\rangle \cdot \left\langle \frac{5}{13}, \frac{12}{13} \right\rangle$$

$$= \frac{-20-120}{1300} = \frac{-14\cancel{0}}{130\cancel{0}} = \frac{-14}{130}$$

CamScanner ile tarandı

Now, we'll determine the direction $\vec{v}$ at $(3,0)$ in which the rate of change of f is the largest

So, $\vec{v}$ and $\vec{\nabla} f(3,0)$ have the same direction.

$$\vec{\nabla} f(3,0) = \left\langle \frac{-2\cdot 3}{(1+9+0)^2}, \frac{-10\cdot 0}{(1+9+0)^2} \right\rangle$$

$$= \left\langle -\frac{6}{100}, 0 \right\rangle \quad \text{(not unit vector)}$$

$$\vec{v} = \frac{\vec{\nabla} f(3,0)}{\|\vec{\nabla} f(3,0)\|} = \langle -1, 0 \rangle \quad \text{(Now, it is a unit vector)}$$

The largest rate of change of f is

$$D_{\vec{v}} f(3,0) = \left\langle -\frac{6}{100}, 0 \right\rangle \cdot \langle -1, 0 \rangle$$

$$= \frac{6}{100}.$$

CamScanner ile tarandı

⑤ Calculate $\frac{\partial^2 y}{\partial x \partial z}$ if x, y, z satisfy the equation

$$x^3y + y^3z + z^3x = 1.$$

Soln: Say $F(x,y,z) = x^3y + y^3z + z^3x - 1.$

If $\frac{\partial F}{\partial y} = x^3 + 3y^2z \neq 0$, then y can be considered as a function of x and z by the implicit func. thm.

$$\frac{\partial^2 y}{\partial x \partial z} = \frac{\partial}{\partial x}\left(\frac{\partial y}{\partial z}\right)$$

$$\frac{\partial (x^3y + y^3z + z^3x)}{\partial z} = \frac{\partial (1)}{\partial z}$$

$$\Rightarrow x^3 \cdot \frac{\partial y}{\partial z} + 3 \cdot y^2 \cdot \frac{\partial y}{\partial z} \cdot z + y^3 \cdot 1 + 3 \cdot z^2 \cdot x = 0$$

$$\Rightarrow \frac{\partial y}{\partial z}\left[x^3 + 3zy^2\right] = -3xz^2 - y^3$$

$$\Rightarrow \frac{\partial y}{\partial z} = \frac{-3xz^2 - y^3}{x^3 + 3zy^2}.$$

$$\frac{\partial}{\partial x}\left(\frac{\partial y}{\partial z}\right) = \frac{\partial}{\partial x}\left(\frac{-3xz^2 - y^3}{x^3 + 3zy^2}\right)$$

$(-3xz^2 - y^3) \cdot (3x^2 + 6zy \cdot \frac{\partial y}{\partial x})$

(b) If $e^{yz} - x^2 z \ln y = \pi$, calculate $\frac{\partial y}{\partial z}$ and provide the condition which guarantees the existence of solution.

Soln:

$$\frac{\partial (e^{yz} - x^2 z \ln y)}{\partial z} = \frac{\partial (\pi)}{\partial z}$$

$$\Rightarrow \quad e^{yz}\left(z\frac{\partial y}{\partial z} + y\right) - \left(x^2 \ln y + x^2 z \frac{\frac{\partial y}{\partial z}}{y}\right) = 0$$

$$\Rightarrow \quad \frac{\partial y}{\partial z}\left[z e^{yz} - \frac{x^2 z}{y}\right] = x^2 \ln y - y e^{yz}$$

$$\Rightarrow \quad \frac{\partial y}{\partial z} = \frac{x^2 \ln y - y e^{yz}}{z e^{yz} - \frac{x^2 z}{y}}$$

Say $F(x,y,z) = e^{yz} - x^2 z \ln y - \pi$.

$\frac{\partial F}{\partial y} = z e^{yz} - \frac{x^2 z}{y} \neq 0$, is the condition which guarantees the existence of solution by the Implicit function theorem

7) Show that the equation $xy + z^3x - 2yz = 0$ defines z as a function of x and y, (i.e, $z = z(x,y)$) near the point $(0,1)$ with $z(0,1) = 0$. Calculate $\frac{\partial z}{\partial x}$ at $(0,1)$.

Soln :

Say $F(x,y,z) = xy + z^3x - 2yz$.

$$\left.\frac{\partial F}{\partial z}\right|_{(0,1)} = 3z^2x - 2y\Big|_{(0,1)} = -2 \neq 0.$$

By the implicit function theorem, z can be considered as a function of x and y near $(0,1)$. Also, Put $x=0$, $y=1$. $0 + z^3 \cdot 0 - 2 \cdot 1 \cdot z = 0$

$\Rightarrow z = 0.$

$\Rightarrow z(0,1) = 0.$

$$\frac{\partial (xy + z^3x - 2yz)}{\partial x} = 0$$

$$y + 3 \cdot z^2 \cdot x \frac{\partial z}{\partial x} + z^3 - 2y\frac{\partial z}{\partial x} = 0$$

$$\frac{\partial z}{\partial x}\left[3z^2x - 2y\right] = -z^3 - y$$

$$\Rightarrow \frac{\partial z}{\partial x} = \frac{-z^3 - y}{3z^2x - 2y}. \quad \left.\frac{\partial z}{\partial x}\right|_{(0,1)} = \frac{-(z(0,1))^3 - 1}{3(z(0,1))^2 \cdot 0 - 2 \cdot 1} = \frac{-1}{-2} = \frac{1}{2}$$

(8) Find and Classify the critical points of the function $f(x,y) = x^4 + y^4 + 4xy$.

Soln:

Defn: (Critical pts)

A point (a,b) is called a critical pt of f if

$$f_x(a,b) = f_y(a,b) = 0$$

$Dom f = \mathbb{R}^2$

$f_x = 4x^3 + 4y$

$f_y = 4y^3 + 4x$

If (a,b) is a critical pt, then

$f_x(a,b) = 4a^3 + 4b = 0$ & $f_y(a,b) = 4b^3 + 4a = 0$

$\Downarrow$

$b = -a^3 \Rightarrow 4(-a^3)^3 + 4a = 0$

$\Rightarrow -a^9 + a = 0$

$\Rightarrow a(-a^8 + 1) = 0$

$\Rightarrow a = 0$ OR $a = \mp 1$

<u>Recall</u> The second derivative test

Let $f_{xx}, f_{xy}, f_{yx}, f_{yy}$ be continuous on a disk with center (a,b). Let (a,b) be a critical pt of f and

$$D = \begin{vmatrix} f_{xx}(a,b) & f_{xy}(a,b) \\ f_{yx}(a,b) & f_{yy}(a,b) \end{vmatrix} = f_{xx}(a,b)f_{yy}(a,b) - [f_{xy}(a,b)]^2$$

(by mixed partial derivative thm, $f_{xy}(a,b) = f_{yx}(a,b)$).

a) If $D > 0$ and $f_{xx}(a,b) > 0$, then $f(a,b)$ is local minimum.

b) If $D > 0$ and $f_{xx}(a,b) < 0$, then $f(a,b)$ is local maximum

c) If $D < 0$, then f has a saddle pt at (a,b)

d) If $D = 0$ then test gives no information.

(i.e. $f(a,b)$ can be local min, local max or (a,b) is a saddle pt.)

$f_{xx} = 12x^2$, $f_{xy} = 4$, $f_{yy} = 12y^2$

For the critical pt $(0,0)$,

$f_{xx}(0,0) = 0$, $f_{xy}(0,0) = 4$, $f_{yy}(0,0) = 0$

$$D = \begin{vmatrix} 0 & 4 \\ 4 & 0 \end{vmatrix} = 0 \cdot 0 - (4)^2 = -16 < 0$$

So, f has a saddle pt at $(0,0)$.

For the critical pt $(1,-1)$,

$f_{xx}(1,-1) = 12$, $f_{xy} = 4$, $f_{yy} = 12$

$$D = \begin{vmatrix} 12 & 4 \\ 4 & 12 \end{vmatrix} = 144 - 16 = 128 > 0.$$

Since $D = 128 > 0$ and $f_{xx}(1,-1) = 12 > 0$, f has a local min. at $(1,-1)$. $f(1,-1) = 1 + 1 - 4 = -2$ is the local min. value.

For the critical pt $(-1,1)$,

$f_{xx}(-1,1) = 12$, $f_{xy}(-1,1) = 4$, $f_{yy}(-1,1) = 12$

$D = 12 \cdot 12 - 4^2 = 128 > 0$

Since $D = 128 > 0$ and $f_{xx}(-1,1) = 12 > 0$, f has a local min. at $\underline{(-1,1)}$. $f(-1,1) = -2$ is the local min. value.