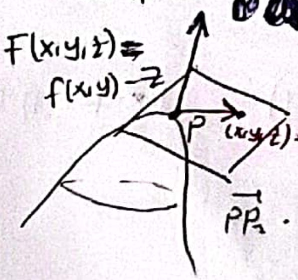
MATH 120 - 2021-2 - RECITATION PROBLEMS

(1) Find the tangent plane to the graph of the function $f(x,y) = xy^3 + x^2$ at the point $(1,3)$. Use a suitable linearization to estimate $f(0.9, 3.1)$



soln:

A normal vector to $z = f(x,y)$ at $(1, 3, f(1,3))$ is $\vec{n} = \langle f_x(1,3),\ f_y(1,3),\ -1 \rangle.$

$f(1,3) = 1 \cdot 3^3 + 1^2 = 28$

$f_x(x,y) = y^3 + 2x \Rightarrow f_x(1,3) = 27 + 2 = 29$

$f_y(x,y) = 3xy^2 \Rightarrow f_y(1,3) = 3 \cdot 1 \cdot 3^2 = 27$

So, $\vec{n} = \langle 29, 27, -1 \rangle$

An equation of the tangent plane to $z = f(x,y)$ at $(1, 3, 28)$ is

$$z = f(1,3) + f_x(1,3)(x-1) + f_y(1,3)(y-3)$$

$$\Leftrightarrow \boxed{z = 28 + 29(x-1) + 27 \cdot (y-3)}$$

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To estimate $f(0.9, 3.1)$, we need to use the linearization of f at $(1,3)$. We have observed the linearization of f at $(1,3)$ which is tangent plane of f at $(1,3)$:

$$L(x,y) = 28 + 29(x-1) + 27.(y-3).$$

Then,

$$f(x,y) \approx L(x,y) = 28 + 29(x-1) + 27(y-3)$$

$$f(0.9, 3.1) \approx 28 + 29(0.9-1) + 27(3.1-3)$$

$$= 28 + 29(-0.1) + 27(0.1)$$

$$= 28 - 2.9 + 2.7$$

$$= 28 - 0.2 = 28 - \frac{1}{5} = \frac{139}{5}$$

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(2) What horizontal plane is tangent to the surface

$z = f(x, y) = x^2 - 4xy - 2y^2 + 12x - 12y - 1$?

Soln :

A plane is horizontal only if its equation is of the form $z = k$, i.e. it is independent of x and y.

Thus, $\frac{\partial z}{\partial x} = \frac{\partial z}{\partial y} = 0$ at the point of tangency

$$\frac{\partial z}{\partial x} = \frac{\partial f}{\partial x} = 2x - 4y + 12 = 0 \Rightarrow x - 2y = -6$$

$$\frac{\partial z}{\partial y} = \frac{\partial f}{\partial y} = -4x - 4y - 12 = 0 \Rightarrow x + y = -3$$

$$\Rightarrow 3y = 3 \Rightarrow \boxed{y = 1}$$

$$\Rightarrow \boxed{x = -4}$$

So, $z = f(-4, 1) = 16 + 16 - 2 - 48 - 12 - 1 = -31$

Thus, the required tangent plane has equation

$z = -31$ at $(-4, 1, -31)$.

(3) Write the equation of the tangent plane to the surface $xz + 2x^2y - yz^2 = -11$ at the point $(1,2,3)$.

Soln :

Since the gradient vector is a normal vector for the tangent plane at $(1,2,3)$.

$$\vec{\nabla} f(x,y,z) = \langle f_x(x,y,z),\ f_y(x,y,z),\ f_z(x,y,z) \rangle$$

$$= \langle z + 4xy,\ 2x^2 - z^2,\ x - 2yz \rangle$$

where $f(x,y,z) = xz + 2x^2y - yz^2 + 11$

$$\vec{n} = \vec{\nabla} f(1,2,3) = \langle 3 + 4.1.2,\ 2.1^2 - 3^2,\ 1 - 2.2.3 \rangle$$

$$= \langle 11,\ -7,\ -11 \rangle$$

So, the equation of the tangent plane to f at $(1,2,3)$ is

$$\vec{\nabla} f \cdot \langle x-1,\ y-2,\ z-3 \rangle = 0$$

$$\Leftrightarrow \langle 11, -7, -11 \rangle \cdot \langle x-1,\ y-2,\ z-3 \rangle = 0$$

$$\Leftrightarrow 11x - 11 - 7x + 14 - 11z + 33 = 0$$

$$\Leftrightarrow \boxed{11x - 7x - 11z = -36}$$

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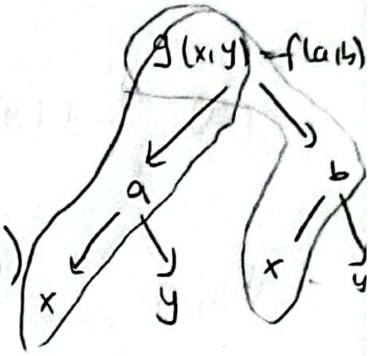
④ Let $g(x,y) = f(x^2 + f(x,y), f(f(x,y)), y)$ for a differentiable function f with $f(1,1) = 1$, $f_1(1,1) = -1$, $f_2(1,1) = 3$, $f_1(2,1) = 2$ and $f_2(2,1) = -2$. Find $g_1(1,1)$.



<u>Soln</u>:

$$g(x,y) = f(\underbrace{x^2 + f(x,y)}_{a}, \underbrace{f(f(x,y)), y}_{b}))$$

$$g_1(x,y) = f_1(x^2 + f(x,y), f(f(x,y)), y)) \cdot (2x + f_1(x,y))$$

$$+ f_2(x^2 + f(x,y), f(f(x,y),y)) \cdot [f_1(f(x,y),y) \cdot (f_1(x,y))$$

$$f_2(f(x,y),y) \cdot 0]$$

$$g_1(1,1) = f_1(1^2 + \underbrace{f(1,1)}_{=1}, f(\underbrace{f(1,1)}_{=1}, 1)) \cdot (2 \cdot 1 + \underbrace{f_1(1,1)}_{=-1})$$

$$+ f_2(1^2 + \underbrace{f(1,1)}_{=1}, f(\underbrace{f(1,1)}_{=1},1)) [f_1(\underbrace{f(1,1)}_{=1},1) \cdot \underbrace{f_1(1,1)}_{=-1}]$$

$$= f_1(2, \underbrace{f(1,1)}_{=1}) + f_2(2, \underbrace{f(1,1)}_{=1}) [\underbrace{f_1(1,1)}_{=-1} \cdot (-1)]$$

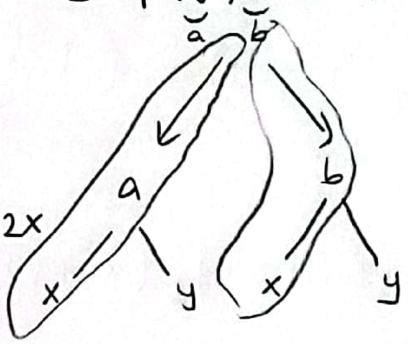
$$= f_1(2,1) + f_2(2,1)$$

$$= 2 - 2 = 0.$$

⑤ Given that f is a differentiable function with continuous partial derivatives; calculate the following in terms of the partial derivatives of f

(a) $\frac{\partial}{\partial x} f(y^5, x^2)$

$z = f(\underbrace{y^5}_{a}, \underbrace{x^2}_{b}) = f(a, b)$



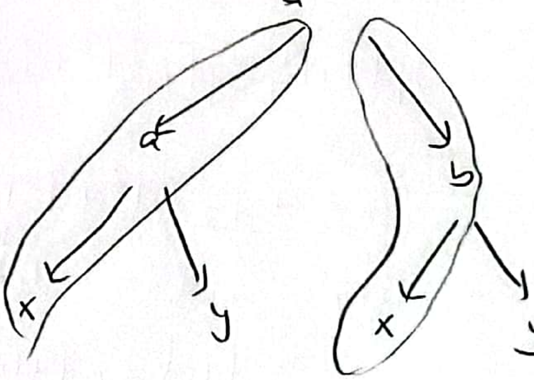
<u>Soln</u>:

$$\frac{\partial f(y^5, x^2)}{\partial x} = f_1(y^5, x^2) \cdot 0 + f_2(y^5, x^2) \cdot 2x$$

$$= 2x \cdot f_2(y^5, x^2).$$

(b) $\frac{\partial}{\partial x} f(y^2 f(x, y^2), f(x^2, y))$

$z = f(\underbrace{y^2 f(x, y^2)}_{a}, \underbrace{f(x^2, y)}_{b})$



<u>Soln</u>:

$$\frac{\partial f(y^2 f(x,y^2), f(x^2, y))}{\partial x} = f_1(y^2 f(x,y^2), f(x^2,y))$$

$$\cdot \left[y^2 \cdot f_1(x, y^2) \cdot 1 + y^2 \cdot f_2(x, y^2) \cdot 0 \right] + f_2(y^2 f(x,y^2), f(x^2,y))$$

$$\cdot \left[f_1(x^2, y) \cdot 2x + f_2(x^2, y) \cdot 0 \right]$$

$$= f_1(y^2 f(x,y^2), f(x^2,y)) \cdot y^2 f_1(x, y^2) + f_2(y^2 f(x,y^2), f(x^2,y))$$

$$\cdot 2x f_1(x^2, y)$$

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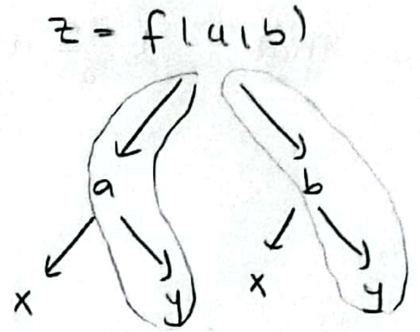
c) $\frac{\partial}{\partial y} f(\underbrace{x^2 f(x+y^3, yx)}_{a}, \underbrace{f(x, yx^2)}_{b})$

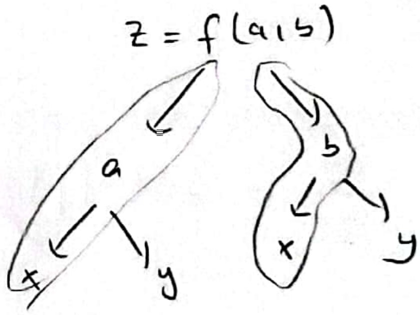
Soln:



$$\frac{\partial}{\partial y} f(x^2 f(x+y^3, yx), f(x, yx^2))$$

$$= f_1(x^2 f(x+y^3, yx), f(x, yx^2)) \cdot [x^2 [f_1(x+y^3, yx) . 3y^2 + f_2(x+y^3, yx) \cdot x] + f_2(x^2 f(x+y^3, yx), f(x, yx^2)) \cdot [f_1(x, yx^2) . 0 + f_2(x, yx^2) . x^2]$$

d) $\frac{\partial^2}{\partial y \partial x} f(x^3, xy+y^2)$



Soln:

$$\frac{\partial}{\partial y}\left(\frac{\partial}{\partial x} f(\underbrace{x^3}_{a}, \underbrace{xy+y^2}_{b})\right) = ?$$

$$\frac{\partial}{\partial x} f(x^3, xy+y^2) = f_1(x^3, xy+y^2) . 3x^2 + f_2(x^3, xy+y^2) . y$$

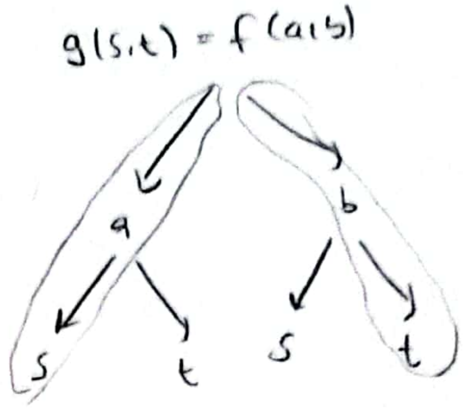
$$\frac{\partial}{\partial y}\left(3x^2 f_1(x^3, xy+y^2) + y . f_2(x^3, xy+y^2)\right) = 3x^2 [f_{12}(x^3, xy+$$

$$\cdot (x+2y)] + 1 \cdot f_2(x^3, xy+y^2) + y . [f_{22}(x^3, xy+y^2) . (x+2y)$$

⑥ If $g(s,t) = f(s^2-t^2,\ t^2-s^2)$ and f is differentiable,

Show that $t\dfrac{\partial g}{\partial s} + s\dfrac{\partial y}{\partial t} = 0$.

<u>Soln</u>



$$\frac{\partial g}{\partial s} = f_1(s^2-t^2,\ t^2-s^2)\cdot 2s + f_2(s^2-t^2, t^2-s^2)\cdot(-2s)$$

$$\frac{\partial g}{\partial t} = f_1(s^2-t^2, t^2-s^2)\cdot(-2t) + f_2(s^2-t^2, t^2-s^2)\cdot(2t)$$

$$t\ \frac{\partial g}{\partial s} = 2st\ f_1(s^2-t^2, t^2-s^2) - 2st\ f_2(s^2-t^2, t^2-s^2)$$

$$s\cdot\frac{\partial y}{\partial t} = -2st\ f_1(s^2-t^2, t^2-s^2) + 2st\ f_2(s^2-t^2,\ t^2-s^2)$$

$$t\cdot\frac{\partial g}{\partial s} + s\cdot\frac{\partial y}{\partial t} = 0$$

⑦ Calculate $D_u(D_u(2x^2+3xy+y^2))$ where D_u denotes the directional derivative in the direction $u = \langle 1,-1 \rangle$.

Soln :

Let $f(x,y) = 2x^2+3xy+y^2$ be

If f is diff'ble, $D_{\vec{u}} f(P_0) = \vec{\nabla} f(P_0) \cdot \frac{\vec{u}}{\|\vec{u}\|}$.

Theorem: If f_1 and f_2 are continuous in a nbd of the pt (a,b), then f is diff'ble at (a,b)

$f_1(x,y) = 4x+3y$ & $f_2(x,y) = 3x+2y$

are continuous everywhere since they are polynomials.

So, f is diff'ble everywhere.

$$D_{\vec{u}} f(x,y) = \vec{\nabla} f(x,y) \cdot \frac{\langle 1,-1 \rangle}{\sqrt{2}}$$

$$= \langle 4x+3y, 3x+2y \rangle \cdot \langle \tfrac{1}{\sqrt{2}}, -\tfrac{1}{\sqrt{2}} \rangle$$

$$= \frac{4x+3y}{\sqrt{2}} - \frac{3x+2y}{\sqrt{2}} = \frac{x}{\sqrt{2}} + \frac{y}{\sqrt{2}}$$

$$D_{\vec{u}} f(x,y) = \frac{1}{\sqrt{2}}(x+y)$$

Similarly, $D_{\vec{u}} f(x,y)$ is diff'ble everywhere.

$$D_{\vec{u}}(D_{\vec{u}} f(x,y)) = \vec{\nabla}(D_{\vec{u}} f(x,y)) \cdot \frac{\vec{u}}{\|\vec{u}\|}$$

$$= \left\langle \frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}} \right\rangle \cdot \left\langle \frac{1}{\sqrt{2}}, \frac{-1}{\sqrt{2}} \right\rangle$$

$$= \frac{1}{2} - \frac{1}{2} = 0.$$