Week 7

1) Find the limits if they exist

(a) $\lim_{(x,y)\to(0,0)} (x^2+y^2)^{1/2} \sin\left(\frac{1}{x^2+y^2}\right)$

Soln| $\text{Dom}(f) = \mathbb{R}^2 \setminus \{(0,0)\}$

$$\sqrt{x^2+y^2} \geq 0$$

$\forall (x,y) \neq (0,0)$, $-1 \leq \sin\left(\frac{1}{x^2+y^2}\right) \leq 1 \Rightarrow -\sqrt{x^2+y^2} \leq \sqrt{x^2+y^2}\sin\left(\frac{1}{x^2+y^2}\right) \leq \sqrt{x^2+y^2}$

Since $\lim_{(x,y)\to(0,0)} -\sqrt{x^2+y^2} = 0 = \lim_{(x,y)\to(0,0)} \sqrt{x^2+y^2}$, by Squeeze thm, $\lim_{(x,y)\to(0,0)} f(x,y) = 0$

(b) $\lim_{(x,y)\to(1,0)} \frac{(x-1)^3 y}{(x-1)^6 + y^2}$

Soln.: $\text{Dom}(f) = \mathbb{R}^2 \setminus \{(1,0)\}$

along $y = x-1$

$$\lim_{x\to 1} \frac{(x-1)^4}{(x-1)^6 + (x-1)^2} = \lim_{x\to 1} \frac{(x-1)^2}{(x-1)^4 + 1} = 0$$

$\not\equiv$

along $y = (x-1)^3$

$$\lim_{x\to 1} \frac{(x-1)^6}{(x-1)^6 + (x-1)^6} = \lim_{x\to 1}\left(\frac{1}{2}\right) = \frac{1}{2}$$

$\Bigg\} \lim_{(x,y)\to(1,0)} \frac{(x-1)^3 y}{(x-1)^6 + y^2} = \text{d.n.e.}$

(c) $\lim_{(x,y)\to(0,0)} \frac{x^2 y^2}{x^2+y^4}$

Soln| $\text{Dom}(f) = \mathbb{R}^2 \setminus \{(0,0)\}$

$$\underbrace{0}_{\to 0 \text{ as } (x,y)\to(0,0)} \leq \frac{x^2}{x^2+y^4} \cdot y^2 \leq \underbrace{1 \cdot y^2}_{\to 0 \text{ as } (x,y)\to(0,0)}$$

Thus, by Squeeze thm, $\lim_{(x,y)\to(0,0)} \frac{x^2y^2}{x^2+y^4} = 0$

(d) $\lim_{(x,y)\to(0,1)} \frac{\sin\left(\pi\left[x^2+(y-1)^2\right]\right)}{|x| + |y-1|}$

Soln)

$$\lim_{(x,y)\to(0,1)} \frac{\sin(\pi[x^2+(y-1)^2])}{\pi[x^2+(y-1)^2]} \cdot \frac{\pi[x^2+(y-1)^2]}{|x|+|y-1|} = I$$

$$\lim_{(x,y)\to(0,1)} \frac{\sin(\pi[x^2+(y-1)^2])}{\pi[x^2+(y-1)^2]} = 1$$ & Observe that $x^2=|x||x|$ & $(y-1)^2=|y-1||y-1|$

$$\cancel{0} \leq \frac{|x|\cdot|x|}{|x|+|y-1|} \leq 1\cdot\cancel{|x|}$$ as $(x,y)\to(0,1)$ $\to 0$, as $(x,y)\to(0,1)$ $\Rightarrow$ By Squeeze thm, $\lim_{(x,y)\to(0,1)} \frac{x^2}{|x|+|y-1|}=0$

$$\cancel{0} \leq \frac{|y-1|\cdot|y-1|}{|x|+|y-1|} \leq 1\cdot\cancel{|y-1|}$$ as $(x,y)\to(0,1)$ $\to 0$, as $(x,y)\to(0,1)$ $\Rightarrow$ By Squeeze thm, $\lim_{(x,y)\to(0,1)} \frac{(y-1)^2}{|x|+|y-1|}=0$

Since each exists

Thus, $I = 1\cdot\pi[0+0]=0$

2) Determine when the following functions are continuous.

(a) $f(x,y) = \sin(x^2+y^2)\, e^{\frac{1}{\cos(x+y)+1}}$

If $\cos(x+y)+1=0$, then it becomes undefined

$\cos(x+y)=-1 \Leftrightarrow x+y=\pi+2k\pi,\ k\in\mathbb{Z} \Rightarrow y=(2k+1)\pi-x,\ k\in\mathbb{Z}$

Thus, $f(x,y)$ is cont. on $\mathbb{R}^2\setminus\{(x,y)\in\mathbb{R}^2 \mid y=(2k+1)\pi-x,\ k\in\mathbb{Z}\}$

(b) $f(x,y)=\begin{cases} 0 & \text{if } (x,y)=(0,0) \\ \frac{\sin(|x|+|y|)}{2(|x|+|y|)} & \text{if } (x,y)\neq(0,0)\end{cases}$

Soln.)

$f(x,y)$ is already cont. for all $(x,y)\in\mathbb{R}^2\setminus\{(0,0)\}$

Continuity at (0,0)

$$\lim_{(x,y)\to(0,0)} \frac{1}{2}\underbrace{\frac{\sin(|x|+|y|)}{(|x|+|y|)}}_{1} = \frac{1}{2} \neq f(0,0)=0 \Rightarrow f(x,y) \text{ is NOT cont. at } (0,0)$$

Thus, $f(x,y)$ is cont. on $\mathbb{R}^2\setminus\{(0,0)\}$

(c) $f(x,y)=\begin{cases} x^2y & \text{if } x<1 \\ y^2(2x-1 & \text{if } x\geq 1\end{cases}$

f is already cont. on $\{(x,y)\in\mathbb{R}^2 \mid x\in(-\infty,1)\cup(1,\infty)\}$

Check the continuity when $x=1$ using $\varepsilon-\delta$ definition (excl)

(3) Evaluate

(a) f_{xy} and f_{yx} where $f(x,y)=xe^{xy}$

$f_{xy}=(f_x)_y$ & $f_{yx}=(f_y)_x$

$f_x=e^{xy}+xye^{xy} \Rightarrow f_{xy}=xe^{xy}+xe^{xy}+x^2ye^{xy}$

$f_y=x^2e^{xy} \Longrightarrow f_{yx}=2xe^{xy}+x^2y\,e^{xy}$

(b) f_{xxy} where $f(x,y)=\underbrace{x^2e^{\sin x}}_{g(x)}+\underbrace{x^2\cos y}_{h(x,y)}$

Soln.)

$\frac{d}{dy}\left(\frac{d^2}{dx^2}g(x)\right)=0$ Thus, $f_{xxy}=0+h_{xxy}$

$h_x=2x\cos y \Rightarrow h_{xx}=2\cos y \Rightarrow h_{xxy}=-2\sin y$

Hence, $f_{xxy}=-2\sin y$

4) Find $\frac{\partial f(0,0)}{\partial x}$ where $f(x,y)=\begin{cases} xy\sin\left(\frac{1}{x^2+y^2}\right) & \text{if } (x,y)\neq(0,0) \\ 0 & \text{if } (x,y)=(0,0)\end{cases}$

Is f_x cont. at $(0,0)$?

Soln)

$$\frac{\partial f(0,0)}{\partial x}=\left.\frac{\partial f(x,y)}{\partial x}\right|_{(x,y)=(0,0)}=\lim_{h\to 0}\frac{f(0+h,0)-f(0,0)}{h}=\lim_{h\to 0}\frac{h\cdot 0\cdot\sin\left(\frac{1}{h^2+0}\right)-0}{h}=$$

$$=\lim_{h\to 0}(0)=0$$

Hence,

$$f_x(x,y)=\begin{cases} y\sin\left(\frac{1}{x^2+y^2}\right)+xy\left(\frac{-2x}{(x^2+y^2)^2}\right)\cos\left(\frac{1}{x^2+y^2}\right) & \text{if } (x,y)\neq(0,0) \\ 0 & \text{if } (x,y)=(0,0)\end{cases}$$

To check continuity at $(0,0)$:

$$\lim_{(x,y)\to(0,0)} f_x(x,y) = \lim_{(x,y)\to(0,0)} \left(y \sin\left(\frac{1}{x^2+y^2}\right) - \frac{2x^2y}{(x^2+y^2)^2} \cos\left(\frac{1}{x^2+y^2}\right)\right)$$

along the curve $y=x$

$$\lim_{x\to 0}\left(x \sin\left(\frac{1}{2x^2}\right) - \frac{2x^3}{4x^4}\cos\left(\frac{1}{2x^2}\right)\right) = \lim_{x\to 0}\left(x \sin\left(\frac{1}{2x^2}\right) - \frac{1}{2x}\cos\left(\frac{1}{2x^2}\right)\right) = \text{dne}$$

along the curve $x=0$

$$\lim_{y\to 0} y \sin\left(\frac{1}{y^2}\right) = 0$$

Thus, even the limit $\lim_{(x,y)\to(0,0)} f_x(x,y) = \text{dne}$

Hence, f_x is <u>not</u> cont. at $(0,0)$