1) Find the area of the indicated region

(a) bounded by the parabola \( x = (y - 1)^2 - 1 \) and the line \( x + y = 1 \)

(b) \( y = 2x^2 \) and \( y = -x^2 + 3x + 6 \)

(a)

\[ x = (y - 1)^2 - 1 \]
\[ x = (y - 1)^2 - 1 = (-x + 1 - 1)^2 - 1 \]
\[ x = (-x)^2 - 1 \rightarrow x^2 - x - 1 = 0 \]
\[ \Delta = 1 - 4(-1).(-1) = 5 \rightarrow x_{1,2} = \frac{-1 \pm \sqrt{5}}{2} = \frac{1}{2} \pm \frac{\sqrt{5}}{2} \]

Area = \( \int \left[ (1 - y) - ((y - 1)^2 - 1) \right] dy = \text{exc!} \)

(b) \( y = 2x^2 \) and \( y = -x^2 + 3x + 6 \)

\[ \Delta = 3^2 - 4(-1)(-6) = 33 \]
\[ x_{1,2} = \frac{-3 \pm \sqrt{33}}{-2} \]

\[ y = 2x^2 = -x^2 + 3x + 6 \]
\[ 3x^2 - 3x - 6 = 0 \]
\[ x^2 - x - 2 = 0 \rightarrow (x - 2)(x + 1) = 0 \]
\[ x = -1 \]

Area = \( \int_{-1}^{2} \left[ (-x^2 + 3x + 6) - (2x^2) \right] dx = \text{exc!} \)

2) Let \( f(x) = 4 - x^2 \) and \( g(x) = 2 + x \). Find the volume of a solid \( S \) obtained by rotating the region enclosed by \( f \) and \( g \) about the x-axis (Write the result as a definite integral, but do NOT evaluate it!)

(No work shown for this part.)
\[
4 - x^2 = 2 + x \Rightarrow x^2 + x - 2 = 0
\]
\[
(x + 2)(x - 1) = 0
\]
\[
x = \pm 1
\]
\[
\text{Area} = A(x) = \pi (R^2 - r^2)
\]
\[
\text{Volume} = \int_{-2}^{2} \pi (R^2 - r^2) \, dx = \frac{1}{2} \pi \left[ (4 - x^2)^2 - (2 + x)^2 \right] \, dx
\]

3) Express the volume \( V(h) \) of water in a 16cm-diameter hemispherical bowl as an integral and, hence, as a function of the depth \( h \) of the water. (Draw a figure!)

**Solution:**
Since the bowl is hemispherical, it can be generated by rotating the quarter-disk (centered at origin with radius \( \frac{16}{2} = 8 \))
\[
-\sqrt{64-x^2} \leq y \leq 0, \quad 0 \leq x \leq 8\] about the y-axis

\[
y = -\sqrt{64-x^2} \Rightarrow x^2 + y^2 = 64 \Rightarrow x = \pm \sqrt{64-y^2} \Rightarrow x = \sqrt{64-y^2}
\]
(Since \( x \) is positive, \( 0 \leq x \leq 8 \))

\[
\text{Volume} = \int_{-8}^{8} \pi r^2 \, dy = \int_{-8}^{8} \pi (64 - y^2) \, dy = \pi \left[ 64y - \frac{y^3}{3} \right]_{y=-8}^{y=h}
\]
4) Let \( S \) be the solid obtained by rotating the finite plane region bounded by the curves \( y = 2 \cos x \), \( y = \cos x \), \( x = 0 \) and \( x = -\frac{\pi}{2} \) about the line \( x = \frac{\pi}{2} \). Write down the integrals (without evaluating) giving the volume of \( S \) by using

(a) Slicing (disk) method
(b) Cylindrical shell method

Solution:

\[
\pi \left[ (64 \cdot h - \frac{h^3}{3}) - (64 \cdot (-8) - \frac{(-8)^3}{3}) \right]
\]

\[ y = 2 \cos x \Rightarrow \frac{y}{2} = \cos x \Rightarrow x = \arccos \left( \frac{y}{2} \right) \]

\[ y = \cos x \Rightarrow x = \arccos (y) \]

(a) \[ A = \pi \left[ \frac{\pi}{2}^{2} - \frac{\pi}{2} \right] \]

\[ r_1 = \arccos \left( \frac{y}{2} \right) + \frac{\pi}{2} \]

\[ r_2 = \arccos \left( \frac{\pi}{2} \right) + \frac{\pi}{2} \]

\[ 0 \leq y \leq 1 \]

\[ 1 \leq y \leq 2 \]

\[ \text{Volume} = \pi \int_0^1 \left[ (\arccos \left( \frac{y}{2} \right) + \frac{\pi}{2})^2 - (\arccos (y) + \frac{\pi}{2})^2 \right] \, dy + \]

\[ + \pi \int_1^2 \left[ (\arccos \left( \frac{\pi}{2} \right) + \frac{\pi}{2})^2 - \left( \frac{\pi}{2} \right)^2 \right] \, dy \]
Volume = \int 2x \cdot \text{ch} \, dx =
\chi = 2x \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} (x + \frac{\pi}{2}) (2\cos x - \cos x) \, dx