Week 9

1) Suppose:

- $f(x)$ is diff'ble
- $f(x) > 0$
- $f(\pi/2) = e^3$
- $f'(\frac{\pi}{2}) = 24$

Find an eqn. for the tangent line to the graph of the function

$$g(x) = [f(x)]^{\cos x}$$

at the point where $x = \frac{\pi}{2}$

Soln

We want to find $g'(\frac{\pi}{2})$.

$$\ln(g(x)) = \ln\left(f(x)^{\cos x}\right) = \cos x \cdot \ln(f(x))$$

$$\frac{g'(x)}{g(x)} = -\sin x \ln(f(x)) + \cos x \cdot \frac{f'(x)}{f(x)}$$

$$\frac{g'(\frac{\pi}{2})}{g(\frac{\pi}{2})} = -\ln\left(f(\tfrac{\pi}{2})\right) \Rightarrow g'(\tfrac{\pi}{2}) = -\underbrace{g(\tfrac{\pi}{2})}_{f(\frac{\pi}{2})^{\cos(\frac{\pi}{2})} = (e^3)^0 = 1} \overbrace{\ln\left(\underbrace{f(\tfrac{\pi}{2})}_{e^3}\right)}^{3} = \underline{\underline{-3}}$$

$$y - 1 = -3\left(x - \frac{\pi}{2}\right)$$

2) Let $f(x)$ be a diff'ble func. with $f(2) = 2$ and $f'(2) = 5$. For $g(x) = f(8 - 3f(x))$ find $g'(2)$.

By Chain Rule, $g'(x) = f'(8 - 3f(x)) \cdot (-3f'(x))$

$$g'(2) = f'(8 - 3f(2)) \cdot (-3f'(2)) = \overbrace{f'(2)}^{5} \cdot (-3 \cdot 5) = \underline{\underline{-75}}$$

3) Find the limit $\lim_{x \to -\infty} \left(\sqrt{x^2 + 10x} - \sqrt{x^2 + 8x}\right)$

$$\lim_{x \to -\infty} \frac{(\cancel{x^2} + 10x) - (\cancel{x^2} + 8x)}{\sqrt{x^2 + 10x} + \sqrt{x^2 + 8x}} = \lim_{x \to -\infty} \frac{2x}{\sqrt{x^2\left(1 + \frac{10}{x}\right)} + \sqrt{x^2\left(1 + \frac{8}{x}\right)}} =$$

$\sqrt{x^2} = |x|$!

$$= \lim_{x\to-\infty} \frac{2x}{|x|\left[\sqrt{1+\frac{10}{x}}+\sqrt{1+\frac{8}{x}}\right]} = \lim_{x\to-\infty} \frac{2\cancel{x}}{-\cancel{x}\left(\sqrt{1+\underbrace{\frac{10}{x}}_{\to 0}}+\sqrt{1+\underbrace{\frac{8}{x}}_{\to 0}}\right)} = -1$$

4) If $\underbrace{\frac{\sin(3x)}{x}}_{\to 3} \leq f(x) \leq \underbrace{3x^2+3}_{\to 3}$ for all $x \neq 0$, find the limit $\lim_{x\to 0} f(x)$

$$\lim_{x\to 0} \underbrace{\frac{\sin(3x)}{3\cdot x}}_{\to 1 \text{ as } x\to 0} \cdot 3 = 3 \qquad \lim_{x\to 0}(3x^2+3) = 3$$

Thus, by Squeeze thm, $\lim_{x\to 0} f(x) = 3$

5) If the func.

$$f(x) = \begin{cases} \dfrac{\sin(kx) - x + x^2\cos\left(\frac{1}{x}\right)}{x} & \text{if } x \neq 0 \\ 7 & \text{if } x = 0 \end{cases}$$

is cont. at $x=0$, find the value of the constant k.

$$\lim_{x\to 0^-} f(x) = \lim_{x\to 0^+} f(x) = f(0) = 7$$

$$\lim_{x\to 0} f(x) = \lim_{x\to 0}\left[\frac{\sin(kx)}{x} - \underbrace{\frac{\cancel{x}}{\cancel{x}}}_{1} + \frac{x^{\cancel{2}}\cos\left(\frac{1}{x}\right)}{\cancel{x}}\right] = \lim_{x\to 0}\left(\frac{\sin(kx)}{x} - 1 + x\cos\left(\frac{1}{x}\right)\right) =$$

$$\lim_{x\to 0} \underbrace{\frac{\sin kx}{kx}}_{\to 1} k = k, \quad \lim_{x\to 0} 1 = 1, \quad -1 \leq \cos\left(\frac{1}{x}\right) \leq 1 \quad \underbrace{-x}_{\to 0 \text{ as } x\to 0} \leq x\cos\left(\frac{1}{x}\right) \leq \underbrace{x}_{\to 0}$$

Thus, by squeeze thm. $\lim_{x\to 0} x\cos\left(\frac{1}{x}\right) = 0$

$$= k - 1 + 0 = k - 1 = 7 \Rightarrow \boxed{k = 8}$$

each limit exists

6) If $\lim_{h\to 0} \dfrac{f'(x+h) - f'(x)}{h} = \cos(5x)$ find $f'''\left(\frac{\pi}{2}\right)$.

Recall: $f'(x) = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h}$

$$f''(x) = \lim_{h \to 0} \frac{f'(x+h) - f'(x)}{h} = \cos(5x) \Rightarrow f'''(x) = -\sin(5x) \cdot 5$$

$$f'''\left(\frac{\pi}{2}\right) = -5$$

7) Let $k > 0$ be a positive constant and let $f(x) = 9x^3 + 9kx - 63$. What is the smallest integer value of k which guarantees that $f(c) = \frac{1}{2}$ for some $c \in (0, 1)$?

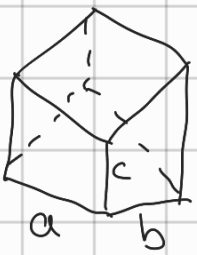
$f(x) = 9x^3 + 9kx - 63$ is a poly. So it is cont. everywhere

$f(0) = -63 < \frac{1}{2}$

$f(1) = 9k - 54$ must be bigger than $\frac{1}{2}$

$9k > \frac{1}{2} + 54 \Rightarrow k > \frac{1}{18} + 6 \Rightarrow \boxed{k = 7}$ is the smallest integer that guarantees $f(c) = \frac{1}{2}$

8) Consider all rectangular prisms (rectangular boxes) which have volume $21600\ cm^3$ and dimensions a, b and c cm s.t. $b = 4a$. Find a so that the surface area of such a rectangular prism is the smallest.



Volume $= a \cdot b \cdot c = 4a^2c = 21600 \Rightarrow \boxed{c = \frac{5400}{a^2}}$

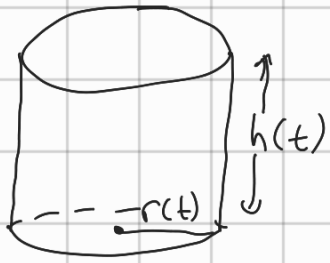
Surface area $= 2(ab + ac + bc) = 2\left(4a^2 + \underbrace{\frac{5400}{a} + \frac{4 \cdot 5400}{a}}_{\frac{5 \cdot 5400}{a}}\right)$

$$S.A.(a) = 8a^2 + \frac{10 \cdot 5400}{a}$$

$$(S.A.(a))' = 16a - \frac{54000}{a^2} = \frac{16a^3 - 54000}{a^2} = 0 \Rightarrow a^3 = 3375$$

$$\boxed{a = 15}$$

9) The radius r of a right circular cylinder decreases at a rate of 0.3 cm/sec and the height h increases at a rate of 0.2 cm/sec. Find the rate of change of the Volume V of the cylinder when $r = 1$ cm and $V = 15\pi$ cm^3. Is V increasing or decreasing?



$r'(t) = -0.3$ cm/sec for all t

$h'(t) = 0.2$ cm/sec for all t

$V(t) = \pi r^2(t)\, h(t)$

when $r = 1$ $V = 15\pi = \pi h$ $\boxed{h = 15}$

By chain Rule

$$V'(t) = 2\pi r(t) r'(t)\, h(t) + \pi r^2(t)\, h'(t)$$

$$V'(t) = 2\pi \cdot 1 \cdot (-0.3)\, 15 + \pi \cdot 1 \cdot (0{,}2)$$

$$= -30\pi \cdot (0{,}3) + 0{,}2\pi = -8{,}8\pi \text{ cm}^3/\text{sec}.$$

decreasing

10) $$\lim_{x \to -\infty} \left(x - \sqrt{x^2 + 4x}\right) = \lim_{x \to -\infty} \frac{\cancel{x^2} - \cancel{x^2} - 4x}{x + \sqrt{x^2 + 4x}} = \lim_{x \to -\infty} \frac{-4x}{x + |x|\sqrt{1 + \frac{4x}{x^2}}}$$

$$= \lim_{x \to -\infty} \frac{-4x}{x - x\sqrt{1 + \frac{4x}{x^2}}} = \lim_{x \to -\infty} \frac{-4\cancel{x}}{\cancel{x}\left(1 - \sqrt{1 + \underbrace{\frac{4\cancel{x}}{\cancel{x^2}}}_{\to 0 \text{ as } x \to -\infty}}\right)} = -\infty$$

$$\frac{4}{x} \xrightarrow[x \to -\infty]{\text{as}} 0^- \Rightarrow \sqrt{1 + \frac{4}{x}} \xrightarrow[x \to -\infty]{\text{as}} 1^- \Rightarrow \left(1 - \sqrt{1 + \frac{4}{x}}\right) \xrightarrow[x \to -\infty]{\text{as}} 0^+$$

$$\lim_{x \to -\infty} x\left(1 + \sqrt{1 + \underbrace{\frac{4}{x}}_{\to 0}}\right) = -\infty \checkmark$$

$y = 2^x$ $y' = ?$

$\ln y = x \ln(2)$

$\frac{y'}{y} = \ln(2) \Rightarrow y' = 2^x \ln(2)$

11) $$\lim_{x \to 0} \frac{2^x - 1}{\sin(4x)} \underset{[L'H]}{\overset{\left[\frac{0}{0}\right]}{=}} \lim_{x \to 0} \frac{\overbrace{2^x}^{1} \ln(2)}{4 \underbrace{\cos(4x)}_{1}} = \frac{\ln(2)}{4}$$

$$\frac{1}{4} \lim_{x \to 0} \frac{4x}{\sin(4x)} \cdot \frac{2^x - 1}{x} = \frac{1}{4} \cdot \lim_{x \to 0} \frac{4x}{\sin(4x)} \cdot \lim_{x \to 0} \frac{2^x - 1}{x} = \frac{1}{4} \cdot 1 \cdot \ln(2)$$

each limit exists

$$\lim_{x\to 0} \frac{4x}{\sin(4x)} = 1, \quad \lim_{x\to 0} \frac{2^x - 2^0}{x-0} = \frac{d}{dx}\left(2^x\right)\Big|_{x=0} = 2^x \ln(2)\Big|_{x=0} = \ln(2)$$

$$\left(f'(x)\Big|_{x=a} = \lim_{x\to a} \frac{f(x)-f(a)}{x-a} \right)$$