1) Suppose:
- \( f(x) \) is differentiable
- \( f(x) > 0 \)
- \( f\left(\frac{\pi}{2}\right) = e^8 \)
- \( f'(\frac{\pi}{2}) = 24 \)

Find an equation for the tangent line to the graph of the function \( g(x) = \left[f(x)\right]^{\cos x} \) at the point where \( x = \frac{\pi}{2} \).

Soh\( l\)n\( 3\):

We want to find \( g'(\frac{\pi}{2}) \).

\[
\ln(g(x)) = \ln\left(f(x)^{\cos x}\right) = \cos x \cdot \ln(f(x))
\]

\[
\frac{g'(x)}{g(x)} = -\sin x \cdot \ln(f(x)) + \cos x \cdot \frac{f'(x)}{f(x)}
\]

\[
\frac{g'(\frac{\pi}{2})}{g\left(\frac{\pi}{2}\right)} = -\ln\left(f\left(\frac{\pi}{2}\right)\right) \Rightarrow g'(\frac{\pi}{2}) = -g\left(\frac{\pi}{2}\right) \ln\left(f\left(\frac{\pi}{2}\right)\right) = -3
\]

\[
f\left(\frac{\pi}{2}\right) = (e^8)^{\cos\left(\frac{\pi}{2}\right)} = (e^8)^0 = 1
\]

\[
y = -3 \left(x - \frac{\pi}{2}\right)
\]

2) Let \( f(x) \) be a differentiable function with \( f(2) = 2 \) and \( f'(2) = 5 \). For \( g(x) = f(8 - 3f(x)) \) find \( g'(2) \).

By the Chain Rule, \( g'(x) = f'(8 - 3f(x)) \cdot (-3f'(x)) \)

\[
g'(2) = f'(8 - 3f(2)) \cdot (-3f'(2)) = f'(2) \cdot (-3 \cdot 5) = -25
\]

3) Find the limit \( \lim_{x \to \infty} \left(\sqrt{x^2 + 10x} - \sqrt{x^2 + 8x}\right) \).

\[
\lim_{x \to \infty} \frac{(x^2 + 10x) - (x^2 + 8x)}{\sqrt{x^2 + 10x} + \sqrt{x^2 + 8x}} = \lim_{x \to \infty} \frac{2x}{\sqrt{x^2 + 10x} + \sqrt{x^2 + 8x}}
\]

\[
\lim_{x \to \infty} \left(\sqrt{x^2 + 10x} - \sqrt{x^2 + 8x}\right) = \lim_{x \to \infty} \frac{2x}{2\sqrt{x^2(1 + \frac{10}{x}) + 2\sqrt{x^2(1 + \frac{8}{x})}}}
\]

\[
\sqrt{x^2} = |x| \checkmark
\]
\[
\lim_{x \to -\infty} \frac{2x}{\sqrt{1 + \frac{30}{x}} + \sqrt{1 + \frac{8}{x}}} = \lim_{x \to -\infty} \frac{2x}{\sqrt{10} \cdot \sqrt{1 + \frac{2}{x}}} = -\frac{1}{10}
\]

4) If \( \frac{\sin(3x)}{x} \leq f(x) \leq 3x^2 + 3 \) for all \( x \neq 0 \), find the limit \( \lim_{x \to 0} f(x) \).

\[
\lim_{x \to 3} \frac{\sin(3x)}{3x} \cdot 3 = 3 \quad \lim_{x \to 0} (3x^2 + 3) = 3
\]

Thus, by Squeeze Thm, \( \lim_{x \to 0} f(x) = 3 \).

5) If the function \( f(x) = \begin{cases} \frac{\sin(kx)}{x} + x^2 \cos\left(\frac{1}{x}\right) & \text{if } x \neq 0 \\ 7 & \text{if } x = 0 \end{cases} \) is continuous at \( x = 0 \), find the value of the constant \( k \).

\[
\lim_{x \to 0} f(x) = \lim_{x \to 0} f(x) = f(0) = 7
\]

\[
\lim_{x \to 0} f(x) = \lim_{x \to 0} \left[ \frac{\sin(kx)}{x} \cdot \frac{1}{x} + x^2 \cos\left(\frac{1}{x}\right) \right] = \lim_{x \to 0} \left( \frac{\sin(kx)}{x} - 1 + x \cos\left(\frac{1}{x}\right) \right)
\]

As \( \lim_{x \to 0} \frac{\sin(kx)}{x} = k \) (if \( k \geq 0 \)) and \( \lim_{x \to 0} 1 = 1 \),

\[-1 \leq \cos\left(\frac{1}{x}\right) \leq 1 \quad \Rightarrow \quad -x \leq x \cos\left(\frac{1}{x}\right) \leq x
\]

Thus, by Squeeze Thm, \( \lim_{x \to 0} x \cos\left(\frac{1}{x}\right) = 0 \).

\[
\Rightarrow \quad k - 1 + 0 = k - 1 = 7 \Rightarrow \quad k = 8
\]

Each limit exists.

6) If \( \lim_{h \to 0} \frac{f(x+h) - f(x)}{h} = \cos(5x) \), find \( f''\left(\frac{\pi}{2}\right) \).
Recall: $f'(x) = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h}$

$f''(x) = \lim_{h \to 0} \frac{f'(x+h) - f'(x)}{h} = \cos(5x) \Rightarrow f'''(x) = -\sin(5x) \cdot 5$

$f''(\frac{\pi}{2}) = -5$

7) Let $k > 0$ be a positive constant and let $f(x) = 9x^3 + 9kx - 63$. What is the smallest integer value of $k$ which guarantees that $f(c) = \frac{1}{2}$ for some $c \in (0, 1)$?

$f(x) = 9x^3 + 9kx - 63$ is a poly. So it is cont. everywhere.

$f(0) = -63 < \frac{1}{2}$

$f(1) = 9k - 54$ must be bigger than $\frac{1}{2}$

$9k > \frac{1}{2} + 54 \Rightarrow k > \frac{1}{18} + 6 \Rightarrow \boxed{k = 7}$ is the smallest integer that guarantees $f(c) = \frac{1}{2}$

8) Consider all rectangular prisms (rectangular boxes) which have volume 21600 cm$^3$ and dimensions $a$, $b$ and $c$ cm s.t. $b = 4a$. Find $a$ so that the surface area of such a rectangular prism is the smallest.

Volume $= a \cdot b \cdot c = 4a^2c = 21600 \Rightarrow c = \frac{5400}{a^2}$

Surface area $= 2(ab + ac + bc) = 2\left(4a^2 + \frac{5400}{a} + \frac{4 \cdot 5400}{a}\right)$

$S.A.(a) = 8a^2 + \frac{10 \cdot 5400}{a}$

$(S.A.(a))' = 16a - \frac{54000}{a^2} = \frac{16a^3 - 54000}{a^2} = 0 \Rightarrow a^3 = 3375 \Rightarrow a = 15$
3) The radius \( r \) of a right circular cylinder decreases at a rate of 0.3 \( \text{cm/sec} \) and the height \( h \) increases at a rate of 0.2 \( \text{cm/sec} \). Find the rate of change of the Volume \( V \) of the cylinder when \( r = 1 \text{ cm} \) and \( h = 15\pi \text{ cm}^3 \). Is \( V \) increasing or decreasing?

\[
\begin{align*}
V(t) &= \pi r(t)^2 h(t) \\
\frac{d}{dt} V(t) &= 2\pi r(t)^2 h'(t) + \pi r(t)^2 h(t) \\
&= 2\pi r(t)^2 \cdot (-0.3) + \pi r(t)^2 \cdot 0.2 \\
&= 2\pi r(t)^2 \cdot (-0.1) \\
&= -0.2 \pi \text{ cm}^3/\text{sec}.
\end{align*}
\]

Thus, the rate of change of the volume is negative, indicating that the volume is decreasing.

10) \[
\lim_{x \to -\infty} \left( x - \sqrt{x^2 + 4x} \right) = \lim_{x \to -\infty} \frac{x^2 - x - 4x}{x + \sqrt{x^2 + 4x}} = \lim_{x \to -\infty} \frac{-4x}{1 + \frac{4}{x}} = -4
\]

11) \[
\lim_{x \to 0} \frac{2^x - 1}{\sin(4x)} = \lim_{x \to 0} \frac{2^x \ln(2)}{4 \cos(4x)} = \frac{\ln(2)}{4}
\]

Additionally, for the equation \( y = 2^x \), the derivative is \( y' = x \ln(2) \).
\[
\lim_{x \to 0} \frac{4x}{\sin(4x)} = 1, \quad \lim_{x \to 0} \frac{2^x - 2^0}{x - 0} = \left. \frac{d}{dx} (2^x) \right|_{x=0} = 2^x \ln(2) \bigg|_{x=0} = \ln(2)
\]