Practice Problems: Improper Integrals

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Solutions to the practice problems posted on November 30.

For each of the following problems:

(a) Explain why the integrals are improper.

(b) Decide if the integral is convergent or divergent. If it is convergent, find which value it converges to.

1. $\displaystyle\int_0^\infty \frac{1}{\sqrt[4]{1+x}}\,dx$

   Solution:

   (a) Improper because it is an infinite integral (called a Type I).

   (b) Rewrite:

   $$\int_0^\infty \frac{1}{\sqrt[4]{1+x}}\,dx = \lim_{t\to\infty}\int_0^t \frac{1}{\sqrt[4]{1+x}}\,dx = \lim_{t\to\infty}\int_0^t (1+x)^{-1/4}\,dx = \lim_{t\to\infty}\frac{4}{3}(1+x)^{3/4}\Big|_0^t$$

   $$\lim_{t\to\infty}\frac{4}{3}(1+t)^{3/4} - \frac{4}{3} = \infty$$

   So the integral diverges. □

2. $\displaystyle\int_{-2}^2 \frac{1}{x^2}\,dx$

   Solution: This question was on my subject GRE.

   (a) Improper because the function $\frac{1}{x^2}$ is discontinuous at $x=0$ (called a Type II).

   (b) There are two ways to do this problem, so I will post both solutions.

   One way: Split up the integral at $x=0$:

   $$\int_{-2}^2 \frac{1}{x^2}\,dx = \int_{-2}^0 \frac{1}{x^2}\,dx + \int_0^2 \frac{1}{x^2}\,dx = \lim_{t\to 0^-}\int_{-2}^t \frac{1}{x^2}\,dx + \lim_{s\to 0^+}\int_s^2 \frac{1}{x^2}\,dx$$

   $$= \lim_{t\to 0^-}\frac{-1}{x}\Big|_{-2}^t + \lim_{s\to 0^+}\frac{-1}{x}\Big|_s^2 = \lim_{t\to 0^-}\left(\frac{-1}{t}\right) - \frac{1}{2} - \frac{1}{2} + \lim_{s\to 0^+}\left(\frac{1}{s}\right)$$

   Both of the limits diverge so the integral diverges.

   Another way: $\frac{1}{x^2}$ is an even function, so it is symmetric about $x=0$:

   $$\int_{-2}^2 \frac{1}{x^2}\,dx = 2\int_0^2 \frac{1}{x^2}\,dx = \lim_{t\to 0^+} 2\int_t^2 \frac{1}{x^2}\,dx = \lim_{t\to 0^+} 2\left(\frac{-1}{x}\right)\Big|_t^2 = -1 + 2\lim_{t\to 0^+}\frac{1}{t} = \infty$$

   So the integral diverges. □

3. $\displaystyle\int_{-\infty}^{0} 2^r \; dr$

Solution:

(a) Improper because it is an infinite integral (called a Type I).

(b) Rewrite:

$$\int_{-\infty}^{0} 2^r \; dr = \lim_{t\to-\infty}\int_{t}^{0} 2^r \; dr = \lim_{t\to-\infty}\left(\frac{2^r}{\ln 2}\Big|_t^0\right) = \frac{1}{\ln 2} - \lim_{t\to-\infty}\left(\frac{2^t}{\ln 2}\right) = \frac{1}{\ln 2} - 0 = \frac{1}{\ln 2}$$

Convergent! □

4. $\displaystyle\int_{-\infty}^{\infty} (y^3 - 3y^2) \; dy$

Solution:

(a) Improper because it is an infinite integral (called a Type I).

(b) Need to split it up, try about $y = 0$:

$$\int_{-\infty}^{\infty} (y^3 - 3y^2) \; dy = \int_{-\infty}^{0} (y^3 - 3y^2) \; dy + \int_{0}^{\infty} (y^3 - 3y^2) \; dy$$

$$= \lim_{t\to-\infty}\int_{t}^{0} (y^3 - 3y^2) \; dy + \lim_{s\to\infty}\int_{0}^{s} (y^3 - 3y^2) \; dy = \lim_{t\to-\infty}\left(\frac{y^4}{4} - y^3\right)\Big|_t^0 + \lim_{s\to\infty}\left(\frac{y^4}{4} - y^3\right)\Big|_0^s$$

$$= -\lim_{t\to-\infty}\left(\frac{t^4}{4} - t^3\right) + \lim_{s\to\infty}\left(\frac{s^4}{4} - s^3\right)$$

Both of these limits diverge, so the integral diverges. □

5. $\displaystyle\int_{-\infty}^{\infty} \cos \pi t \; dt$

Solution:

(a) Improper because it is an infinite integral (called a Type I).

(b) Need to split it up, try about $t = 0$:

$$\int_{-\infty}^{\infty} \cos \pi t \; dt = \int_{-\infty}^{0} \cos \pi t \; dt + \int_{0}^{\infty} \cos \pi t \; dt = \lim_{s\to-\infty}\int_{s}^{0} \cos \pi t \; dt + \lim_{r\to\infty}\int_{0}^{r} \cos \pi t \; dt$$

$$= \lim_{s\to-\infty}\left(\frac{1}{\pi}\sin \pi t\right)\Big|_s^0 + \lim_{r\to\infty}\left(\frac{1}{\pi}\sin \pi t\right)\Big|_0^r = -\lim_{s\to-\infty}\left(\frac{1}{\pi}\sin \pi s\right) + \lim_{r\to\infty}\left(\frac{1}{\pi}\sin \pi r\right)$$

Both of these limits diverge, so the integral diverges. □

6. $\displaystyle\int_0^1 \frac{\ln x}{\sqrt{x}}\,dx$

Solution:

(a) Improper because $\frac{\ln x}{\sqrt{x}}$ is undefined at $x = 0$ (called a Type II).

(b) Try a u-substitution first. Let $u = \sqrt{x}$, $du = \frac{1}{2\sqrt{x}}dx \Rightarrow 2du = \frac{1}{\sqrt{x}}dx$. When $x = 0, u = 0$ and when $x = 1, u = 1$:

$$\int_0^1 \frac{\ln x}{\sqrt{x}}\,dx = \int_0^1 \frac{\ln\left(\sqrt{x}^2\right)}{\sqrt{x}}\,dx = \int_0^1 \ln(u^2)\,du = 2\int_0^1 \ln u\,du$$

This is still improper because $\ln u$ is undefined at $u = 0$. Rewrite with a limit:

$$2\int_0^1 \ln u\,du = \lim_{t\to 0^+} 2\int_t^1 \ln u\,du$$

Use integration by parts (we did $\int \ln x dx$ in class once upon a time...):

$$= \lim_{t\to 0^+} 2\left(u\ln u - u\right)\Big|_t^1 = -2 - \lim_{t\to 0^+} 2\left(t\ln t - t\right) = -2 - 2\lim_{t\to 0^+} t\ln t + 2\lim_{t\to 0^+} t$$
$$= -2 - 2\lim_{t\to 0^+} t\ln t + 0 = -2 - 2\lim_{t\to 0^+} t\ln t$$

The right limit is what we call *indeterminate* because if we take the limit we get something that looks like $0\cdot -\infty$, which is no bueno. So we need to use L'Hôpital's Rule (Section 4.4, pg 301 in your textbook):

$$\lim_{t\to 0^+} t\ln t = \lim_{t\to 0^+} \frac{\ln t}{\frac{1}{t}} = \lim_{t\to 0^+} \frac{(\ln t)'}{\left(\frac{1}{t}\right)'} = \lim_{t\to 0^+} \frac{\frac{1}{t}}{\frac{-1}{t^2}} = \lim_{t\to 0^+} \frac{-t^2}{t} = \lim_{t\to 0^+} -t = 0$$

This shows that our integral is convergent, and it converges to $-2 - 2\lim_{t\to 0^+} t\ln t = -2 - 0 = -2$. □

7. $\displaystyle\int_0^\infty \frac{e^x}{e^{2x}+3}\,dx$

Solution:

(a) Improper because it is an infinite integral (called a Type I).

(b) Let's do a u-substitution first. Let $u = e^x$, then $du = e^x dx$. When $x = 0, u = 1$ and when $x \to \infty, u \to \infty$:

$$\int_0^\infty \frac{e^x}{e^{2x}+3}\,dx = \int_0^\infty \frac{e^x}{(e^x)^2+3}\,dx = \int_1^\infty \frac{1}{u^2+3}\,du = \lim_{t\to\infty}\int_1^t \frac{1}{u^2+3}\,du$$
$$= \lim_{t\to\infty} \frac{1}{\sqrt{3}}\tan^{-1}\left(\frac{u}{\sqrt{3}}\right)\Big|_1^t = \lim_{t\to\infty} \frac{1}{\sqrt{3}}\tan^{-1}\left(\frac{t}{\sqrt{3}}\right) - \frac{1}{\sqrt{3}}\tan^{-1}\left(\frac{1}{\sqrt{3}}\right)$$
$$= \frac{1}{\sqrt{3}}\cdot\frac{\pi}{2} - \frac{1}{\sqrt{3}}\cdot\frac{\pi}{6} = \frac{1}{\sqrt{3}}\left(\frac{\pi}{2}-\frac{\pi}{6}\right) = \frac{\pi}{3\sqrt{3}}$$

Convergent! □

8. $\displaystyle\int_0^5 \frac{w}{w-2}\,dw$

Solution:

(a) Improper because the function $\frac{w}{w-2}$ is discontinuous at $w = 2$ (called a Type II).

(b) Try a u-substitution first. Let $u = w-2$, then $w = u+2, du = dw$. When $w = 0, u = -2$, and when $w = 5, u = 3$:

$$\int_0^5 \frac{w}{w-2}\,dw = \int_{-2}^3 \frac{u+2}{u}\,du = \int_{-2}^3 \left(1+\frac{2}{u}\right)\,du$$

This is still a Type II integral since function $1+\frac{2}{u}$ is discontinuous at $u = 0$. Need to split up the integral:

$$\int_{-2}^3 \left(1+\frac{2}{u}\right)\,du = \int_{-2}^0 \left(1+\frac{2}{u}\right)\,du + \int_0^3 \left(1+\frac{2}{u}\right)\,du$$

$$= \lim_{t\to 0^-} \int_{-2}^t \left(1+\frac{2}{u}\right)\,du + \lim_{s\to 0^+} \int_s^3 \left(1+\frac{2}{u}\right)\,du = \lim_{t\to 0^-} (u + 2\ln|u|)\Big|_{-2}^t + \lim_{s\to 0^+} (u + 2\ln|u|)\Big|_s^3$$

$$= \lim_{t\to 0^-} (t + 2\ln|t|) + 2 - 2\ln 2 + 3 + 2\ln 3 - \lim_{s\to 0^+} (s + 2\ln|s|)$$

Both of the limits diverge, so the integral diverges. □

Use the Comparison Theorem to decide if the following integrals are convergent or divergent.

9. $\displaystyle\int_1^\infty \frac{1+e^{-x}}{x}dx$

Solution:

(a) Improper because it is an infinite integral (called a Type I).

(b) Let's guess that this integral is divergent. That means we need to find a function smaller than $\frac{1+e^{-x}}{x}$ that is divergent. To make it smaller, we can make the top smaller or the bottom bigger. Let's make the top smaller:

$$\frac{1+e^{-x}}{x} \geq \frac{1}{x}$$

Then take the integral:

$$\int_1^\infty \frac{1}{x}dx = \lim_{t\to\infty} \int_1^t \frac{1}{x}dx = \lim_{t\to\infty} \ln x\Big|_1^t = \lim_{t\to\infty} \ln t = \infty$$

So the integral diverges. Since $\int_1^\infty \frac{1}{x}dx$ diverges, then $\int_1^\infty \frac{1+e^{-x}}{x}dx$ diverges. □

10. $\displaystyle\int_0^\pi \frac{\sin^2 x}{\sqrt{x}}dx$

Solution:

(a) Improper because the function $\frac{\sin^2 x}{\sqrt{x}}$ is undefined at $x = 0$ (called a Type II).

(b) Let's guess that this integral is convergent. That means we need to find a function bigger than $\frac{\sin^2 x}{\sqrt{x}}$ that is convergent. To make it bigger, we can make the top bigger or the bottom smaller. Let's make the top bigger:

$$\frac{\sin^2 x}{\sqrt{x}} \leq \frac{1}{\sqrt{x}}$$

Then take the integral:

$$\int_0^\pi \frac{1}{\sqrt{x}} dx = \lim_{t \to 0^+} \int_t^\pi \frac{1}{\sqrt{x}} dx = \lim_{t \to 0^+} 2\sqrt{x}\Big|_t^\pi = 2\sqrt{\pi} - \lim_{t \to 0^+} \sqrt{t} = 2\sqrt{\pi} - 0 = 2\sqrt{\pi}$$

So the integral converges. Since $\int_0^\pi \frac{1}{\sqrt{x}} dx$ converges, then $\int_0^\pi \frac{\sin^2 x}{\sqrt{x}} dx$ converges. $\square$