Recitation Week - 08

Q) Sketch the graph of a func. that has the given properties and identify critical points. Assume that f is cont. and its derivatives exist everywhere unless the contrary is implied or explicitly stated.

$f(-1)=0$, $f(0)=2$, $f(1)=1$, $f(2)=0$, $f(3)=1$

$\lim_{x\to\pm\infty}(f(x)-(x-1))=0$, $\left(\Longrightarrow y=x-1 \text{ is an oblique asym.}\right)$

$f'(x)>0$ on $(-\infty,-1)$, $(-1,0)$ and $(2,\infty)$

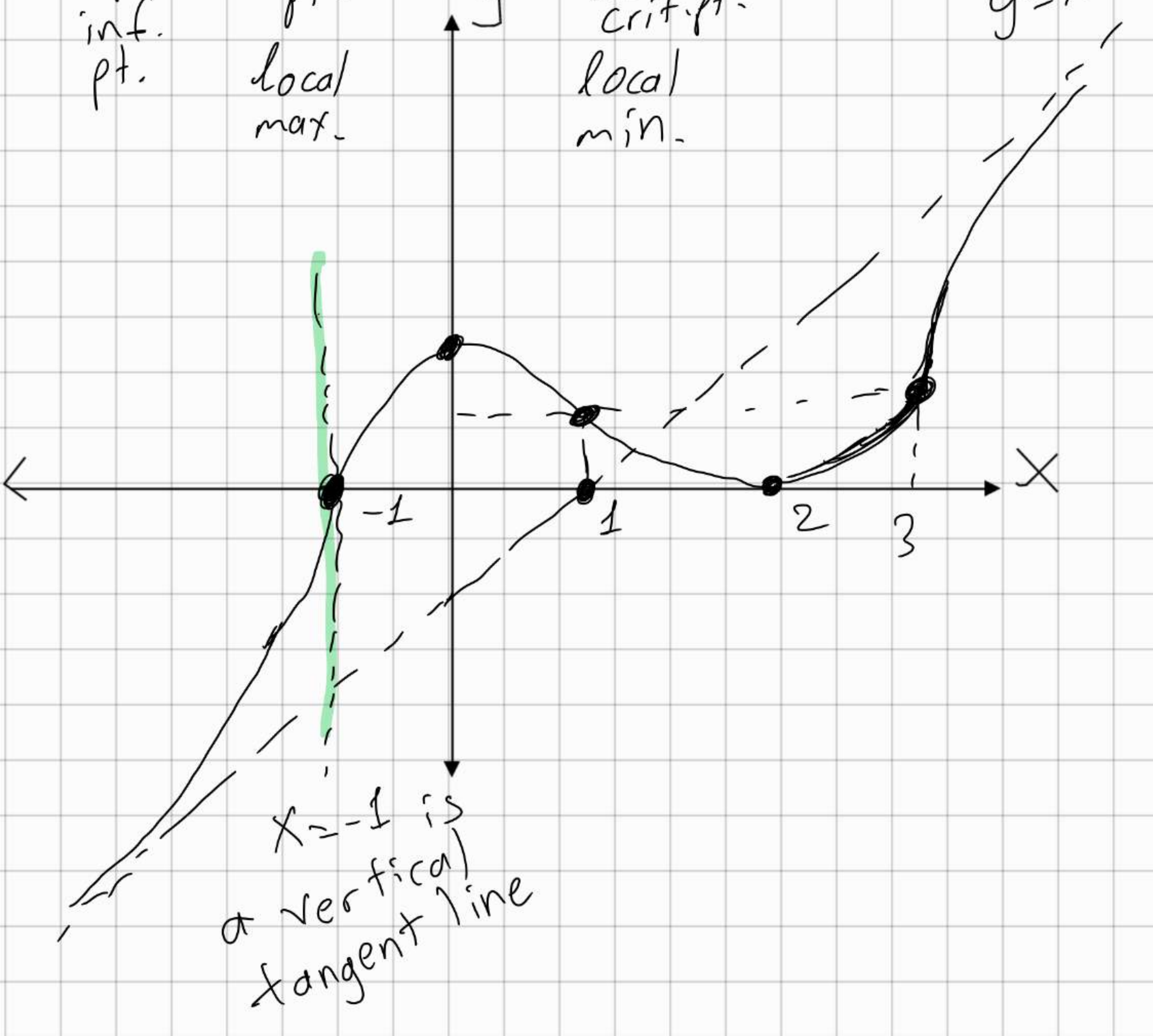
$f'(x)<0$ on $(0,2)$

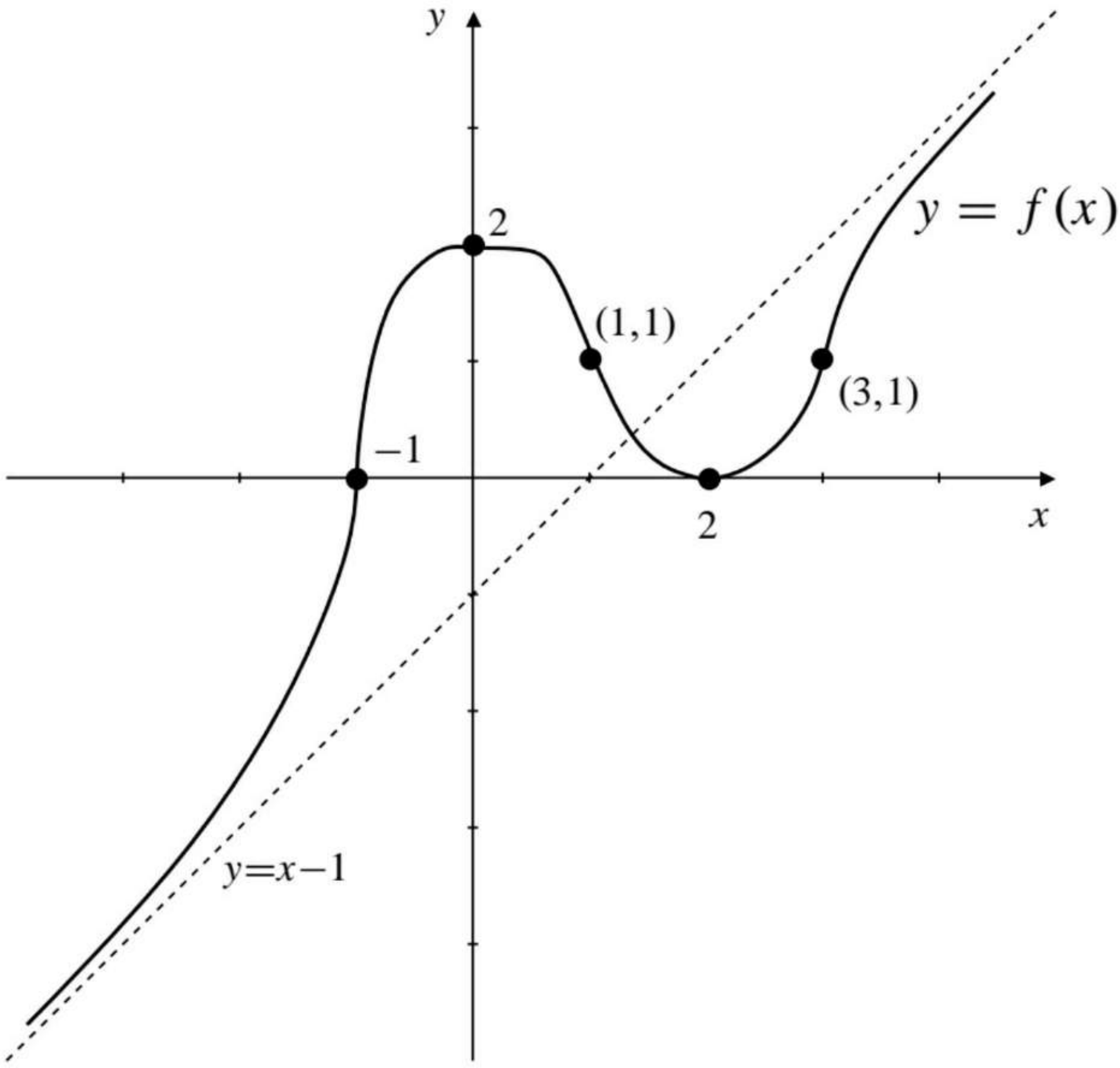
$\lim_{x\to -1} f'(x)=\infty$, $f''(x)>0$ on $(-\infty,-1)$ and $(1,3)$

$\Downarrow$ $f''(x)<0$ on $(-1,1)$ and $(3,\infty)$

$x=-1$ is a singular pt

		-1		0		1		2		3	
$f''(x)$	+	0	−		−	0	+		+	0	−
$f'(x)$	+	‖	+	0	−		−	0	+		+
$f(x)$											
		sing. pt inf. pt.		crit. pt. local max.		inf. pts.		$x=2$ crit. pt. local min.		inf. pts.	





y
2
(1,1)
(3,1)
−1
2
x
$y = f(x)$
$y = x - 1$

1) Sketch the graphs of the following func.s

(a) $f(x) = 2x^{5/3} - 5x^{4/3}$

Solution • $\mathrm{Dom} f = \mathbb{R}$ (domain)

• Intercepts: $x=0 \Rightarrow y=0$

$y=0 \Rightarrow x^{4/3}(2\sqrt[3]{x} - 5) = 0 \Rightarrow x=0, \; x = \left(\frac{5}{2}\right)^3$

roots.

• Asymptotes:

For Horizontal Asym:

$$\lim_{x\to\infty} \sqrt[3]{x^5}\Big(2 - \underbrace{\cancel{5/x^{1/3}}}_{\substack{\text{as } x\to\infty \\ \to 0}}\Big) = \infty, \quad \lim_{x\to-\infty} f(x) = -\infty$$

So, no H.A.

• $f'(x) = \frac{10}{3}x^{2/3} - \frac{20}{3}x^{1/3} = 0 \Rightarrow x=0, \; x=8$

$$\frac{10}{3}x^{1/3}(x^{1/3} - 2) = 0 \Rightarrow \sqrt[3]{x} = 2$$

$$x = 8$$

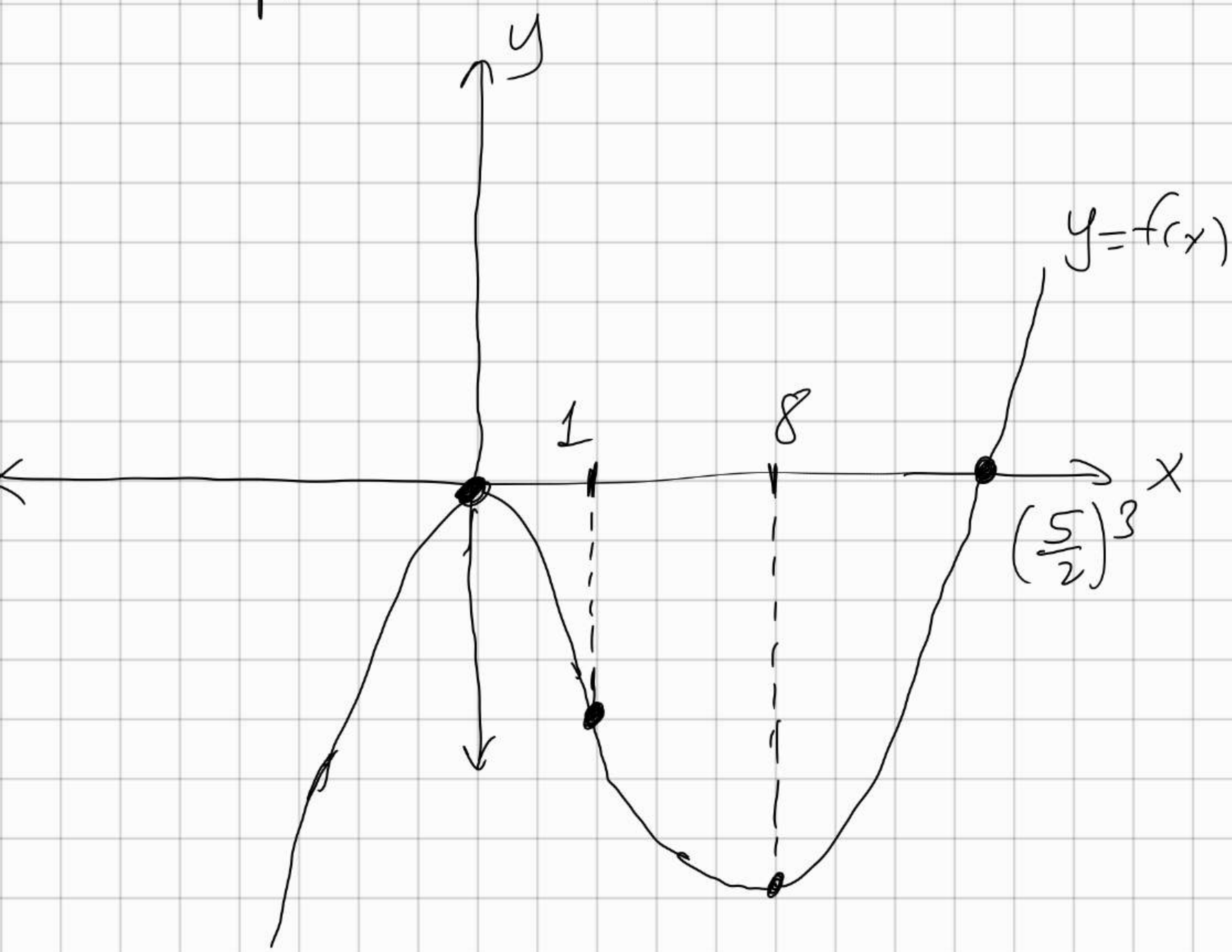
So, $x=0$ and $x=8$ are crit. pts.

• $f''(x) = \frac{20}{9}x^{-1/3} - \frac{20}{9}x^{-2/3} = \frac{20}{9}\left(\frac{1}{\sqrt[3]{x}} - \frac{1}{\sqrt[3]{x^2}}\right)$

$(\sqrt[3]{x})$

$$= \frac{20}{9}\left(\frac{\sqrt[3]{x}-1}{\sqrt[3]{x^2}}\right) \Rightarrow \begin{array}{l} x=1 \Rightarrow f''(1)=0 \\ x=0 \Rightarrow f''(0)=\text{undefined} \end{array}$$

		0		1		8		$(5/2)^3$	
$f''(x)$	$-$	$\Vert$	$-$	0	$+$		$+$		$+$
$f'(x)$	$+$	0	$-$		$-$	0	$+$		$+$
$f(x)$		0						0	
		crit. pt.		inf. pt.		crit. pt.			



(b) $y = f(x) = \dfrac{x^2 - 2x + 2}{x - 1}$

- $\mathrm{Dom} f = \mathbb{R} \setminus \{1\}$

- Interceptts: $x = 0 \Rightarrow y = -2$

$y = 0 \Rightarrow x^2 - 2x + 2 = 0$, $\Delta = 4 - 4 \cdot 2 < 0$

no roots.

- Asymptotes:

$$\lim_{x\to\infty} \frac{x^{\cancel{2}}(1 - \underbrace{2/x}_{\xrightarrow[x\to\infty]{os} 0} + \underbrace{2/x^2}_{\to 0})}{\cancel{x}(1 - \underbrace{1/x}_{\xrightarrow[x\to\infty]{os} 0})} = \infty, \quad \lim_{x\to-\infty} f(x) = -\infty$$

$$f(x) = \frac{x^2 - 2x + 2}{x - 1} = x - 1 + \frac{1}{x-1}$$

$$\begin{array}{r|l} x^2 - 2x + 2 & x - 1 \\ \cline{2-2} -x^2 - x & x - 1 \\ \hline -x + 2 & \\ -\ \ -x + 1 & \\ \hline 1 & \end{array}$$

$(x-1)^2 = x^2 - 2x + 1$

and $\displaystyle\lim_{x\to\mp\infty} \frac{1}{x-1} = 0$

$$\lim_{x\to\pm\infty} \Big(f(x) - (x-1)\Big) = \lim_{x\to\mp\infty} \frac{1}{x-1} = 0$$

So, $y = x - 1$ is an Oblique asymptote.

$$\lim_{x\to 1^-}\left(x-1+\underbrace{\frac{1}{x-1}}\right) = -\infty, \quad \lim_{x\to 1^+} f(x) = \infty$$

$x = 1$ is a V.A.

• Derivatives (increase/decrease & concavity)

$$f'(x) = 1 - \frac{1}{(x-1)^2}, \quad f''(x) = \frac{2}{(x-1)^3}$$

$$\frac{1}{(x-1)^2} = 1 \Rightarrow (x-1)^2 = 1$$

$x=0$, $x=2$ are crit. pts.

$x = 1 \Rightarrow f''(1) =$ undefined

$1 \notin Dom f$

$x = 1$ is <u>not</u> a sing. pt.

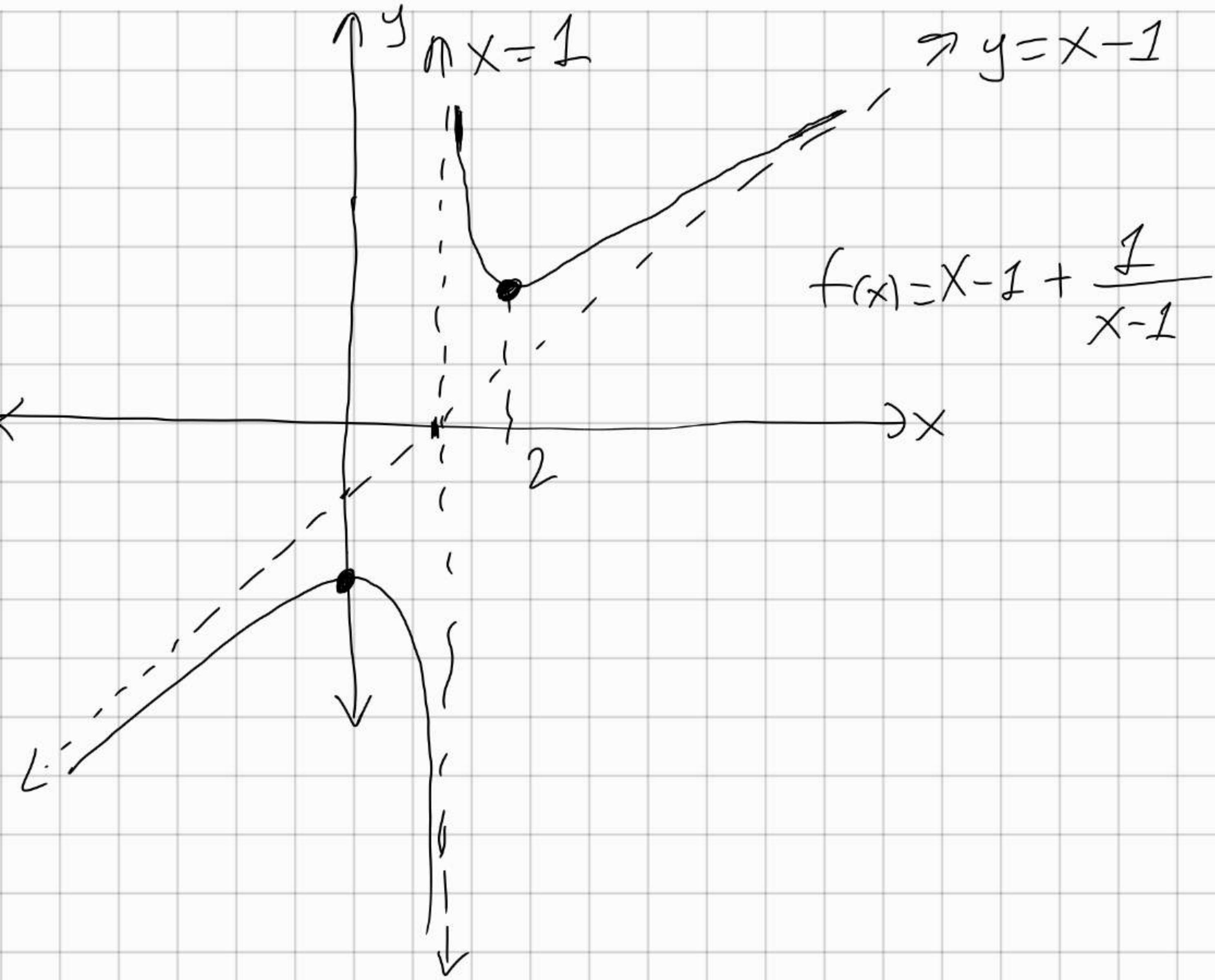
Since $1 \notin Dom f$

		0		1		2	
$f''(x)$	$-$		$-$	‖	$+$		$+$
$f'(x)$	$+$	0	$-$	‖	$-$	0	$+$
$f(x)$				‖			

no inflection pt.

$-1 \le a \le 1 \Rightarrow a^2 \le 1$

$-1 \le x-1 \le 1 \Rightarrow 0 \le x < 2 \Rightarrow f'(x) < 0$



(c) $y = f(x) = \dfrac{x}{\ln(x)}$

- Domain: $\text{Dom} f = \mathbb{R}^+ \setminus \{1\} = (0,1) \cup (1,\infty)$
- Intercepts: $x = 0 \notin \text{Dom} f$, $y = 0 \Rightarrow$ impossible $\Big\}$ no intercepts
- Asymptotes:

$$\lim_{x\to\infty} \frac{x}{\ln(x)} \overset{\left[\frac{\infty}{\infty}\right]}{\underset{[L'H]}{=}} \lim_{x\to\infty} \frac{1}{1/x} = \infty \qquad \text{No H.A.}$$

$\lim_{x \to 1^-} \frac{x}{\ln(x)} = -\infty$, $\lim_{x \to 1^+} \frac{x}{\ln(x)} = \infty \Rightarrow x = 1$ is a V.A.

- $f'(x) = \frac{\ln(x) - 1}{(\ln(x))^2}$

$x = e$ gives a crit. pt.

$x = 1$ is not singular since $\underline{\underline{1 \notin \text{Dom} f}}$

$$f''(x) = \frac{\frac{1}{x}(\ln(x))^2 - (\ln(x) - 1)\frac{2\ln(x)}{x}}{(\ln(x))^4} =$$

$$= \frac{(\ln(x))^2 - 2(\ln(x))^2 + 2\ln(x)}{x(\ln(x))^4}$$

$$= \frac{-\ln(x) + 2}{x(\ln(x))^3} \Rightarrow x = e^2 \Rightarrow f''(x) = 0$$

$x = 0, x = 1 \Rightarrow f''(x) = \text{undefined}$

		0		1		e		e^2	
$f''(x)$			$-$		$+$		$+$	0	$-$
$f'(x)$			$-$		$-$	0	$+$		$+$
$f(x)$									

no inflection

inflection pt.

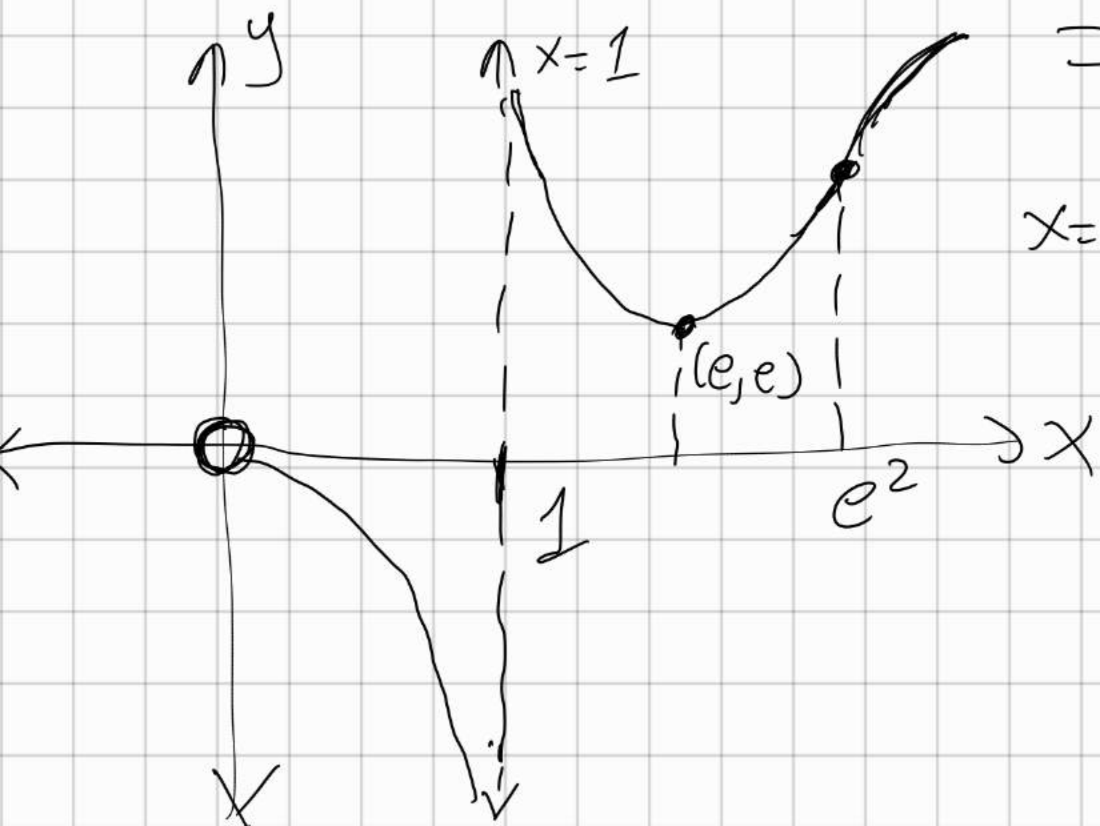
~~$x=0$ into $f(x)=y$~~

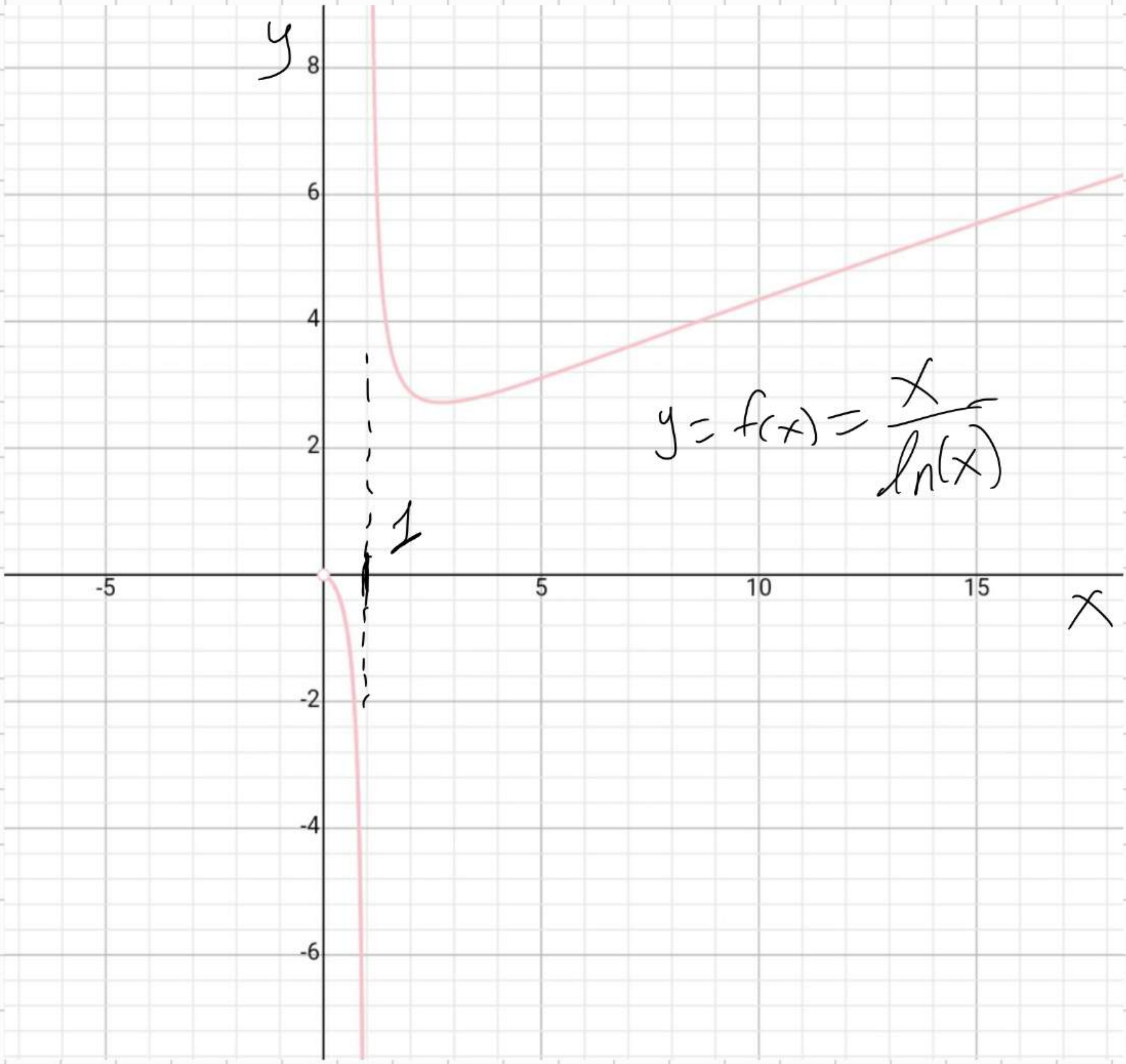
Since each limit exists

$$\lim_{x\to 0^+}\frac{x}{\ln x} = \lim_{x\to 0^+} x \cdot \frac{1}{\ln(x)} = \lim_{x\to 0^+}(x)\cdot\lim_{x\to 0^+}\left(\frac{1}{\ln(x)}\right)$$

$$= 0 \cdot 0 = \underline{\underline{0}}$$

$x=e \Rightarrow f(x)=e$





Note that: Horizontal and oblique asymptotes both can appear at the same time.

ex) $y = \dfrac{xe^x}{1+e^x}$

- Domain: $\text{Dom} f = \mathbb{R}$
- $x = 0$ is a root

• Asymptotes:

$$\lim_{x\to\infty} \frac{xe^x}{1+e^x} = \lim_{x\to\infty} \frac{\cancel{e^x}(x)}{\cancel{e^x}(1/e^x+1)} = \infty$$

$1/e^x \to 0$ as $x \to \infty$

$$\lim_{x\to-\infty} \frac{x}{\cancel{1/e^x+1}} \overset{\left[\frac{\infty}{\infty}\right]}{\underset{L'H}{=}} \lim_{x\to-\infty} \frac{1}{-e^{-x}} = 0 \Rightarrow y=0 \text{ is H. A.}$$

$e^{-x}+1$

Observe that $f(x) = \dfrac{xe^x + x - x}{1+e^x} = x - \dfrac{1}{1+e^x}$

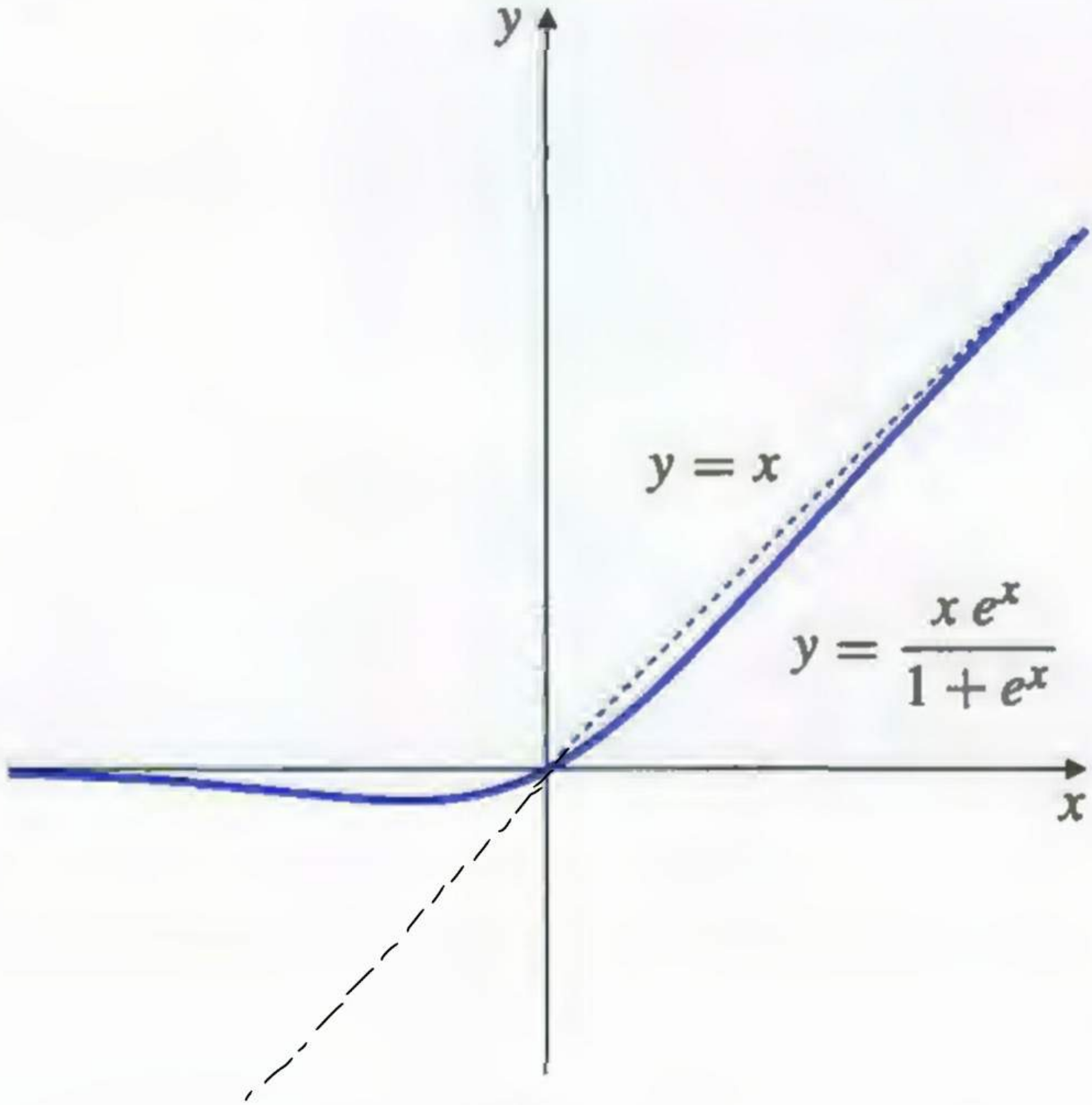
If $\lim\limits_{x\to\mp\infty} \big(f(x) - (ax+b)\big) = 0$, then $y = ax+b$ is an Oblique Asym.

$$\lim_{x\to-\infty} \big(f(x) - x\big) = \lim_{x\to-\infty} \left(\frac{-x}{1+e^x}\right) = \infty$$

So, no oblique asymptote as $x \to -\infty$

$$\lim_{x\to\infty} \big(f(x) - x\big) = \lim_{x\to\infty} \left(\frac{-x}{1+e^x}\right) \overset{\left[\frac{\infty}{\infty}\right]}{\underset{L'H}{=}} \lim_{x\to\infty} \frac{-1}{e^x} = 0$$

So, $y = x$ is oblique asymptote.



y
$y = x$
$y = \dfrac{x\,e^x}{1+e^x}$
x