Practice Problems: Trig Substitution

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The following are solutions to the Trig Substitution practice problems posted on November 9.

1. Use trig substitution to show that $\int \frac{1}{\sqrt{1-x^2}} dx = \sin^{-1} x + C$

   Solution: Let $x = \sin\theta$, then $dx = \cos\theta$:

   $$\int \frac{1}{\sqrt{1-x^2}} dx = \int \frac{1}{\sqrt{1-\sin^2\theta}} \cdot \cos\theta d\theta = \int \frac{\cos\theta}{\cos\theta} d\theta = \int d\theta = \theta + C = \sin^{-1} x + C$$

   □

2. Use trig substitution to show that $\int \frac{1}{1+x^2} dx = \tan^{-1} x + C$

   Solution: Let $x = \tan\theta$, then $dx = \sec^2 x dx$:

   $$\int \frac{1}{1+x^2} dx = \int \frac{1}{1+\tan^2\theta} \cdot \sec^2\theta d\theta = \int \frac{\sec^2\theta}{\sec^2\theta} d\theta = \int d\theta = \theta + C = \tan^{-1} x + C$$

   □

3. $\displaystyle\int \frac{1}{x^2\sqrt{x^2+4}} dx$

   Solution: Let $x = 2\tan\theta$, then $dx = 2\sec^2\theta d\theta$:

   $$\int \frac{1}{x^2\sqrt{x^2+4}} dx = \int \frac{1}{4\tan^2\theta\sqrt{4\tan^2\theta+4}} \cdot 2\sec^2\theta d\theta = \int \frac{2\sec^2\theta}{4\tan^2\theta \cdot 2\sec\theta} d\theta$$

   $$= \int \frac{\sec\theta}{4\tan^2\theta} d\theta = \frac{1}{4}\int \frac{\cos\theta}{\sin^2\theta} d\theta$$

   Now let $w = \sin\theta$, then $dw = \cos\theta d\theta$:

   $$\frac{1}{4}\int \frac{\cos\theta}{\sin^2\theta} d\theta = \frac{1}{4}\int \frac{1}{w^2} dw = -\frac{1}{4w} + C = -\frac{1}{4}\csc\theta + C$$

   Next, we need to plug back in x. Originally we had the substitution $x = 2\tan\theta$, so $\tan\theta = \frac{x}{2}$. This means our opposite side is x, our adjacent side is 2, and the hypotenuse is $\sqrt{x^2+4}$. Then we have

   $$\int \frac{1}{x^2\sqrt{x^2+4}} dx = -\frac{1}{4}\csc\theta + C = -\frac{\sqrt{x^2+4}}{4x} + C$$

   □

4. $\displaystyle\int \frac{\sqrt{1+x^2}}{x}dx$

Solution: Let $x = \tan\theta$, then $dx = \sec^2\theta d\theta$:

$$\int \frac{\sqrt{1+x^2}}{x}dx = \int \frac{\sqrt{1+\tan^2\theta}}{\tan\theta}\cdot\sec^2\theta d\theta = \int \frac{\sec\theta\sec^2\theta}{\tan\theta}d\theta = \int \frac{\sec\theta(1+\tan^2\theta)}{\tan\theta}d\theta$$

$$= \int \left(\frac{\sec\theta}{\tan\theta} + \frac{\sec\theta\tan^2\theta}{\tan\theta}\right)d\theta = \int(\csc\theta + \sec\theta\tan\theta)d\theta = \ln|\csc\theta - \cot\theta| + \sec\theta + C$$

Our substitution was $x = \tan\theta$, so our triangle has opposite side x, adjacent side 1, and hypotenuse $\sqrt{x^2+1}$. Then

$$\int \frac{\sqrt{1+x^2}}{x}dx = \ln|\csc\theta - \cot\theta| + \sec\theta + C = \ln\left|\frac{\sqrt{x^2+1}}{x} - \frac{1}{x}\right| + \sqrt{x^2+1} + C$$

□

5. $\int_0^a x^2\sqrt{a^2-x^2}dx$

Solution: Let $x = a\sin\theta$, then $dx = a\cos\theta d\theta$. We should also change the bounds. When $x = a, \sin\theta = 1 \Rightarrow \theta = \frac{\pi}{2}$. When $x = 0, \sin\theta = 0 \Rightarrow \theta = 0$.

$$\int_0^a x^2\sqrt{a^2-x^2}dx = \int_0^{\pi/2} a^2\sin^2\theta\sqrt{a^2-a^2\sin^2\theta}\cdot a\cos\theta d\theta = a^4\int_0^{\pi/2}\sin^2\theta\cos^2\theta d\theta$$

Use half angle identity:

$$a^4\int_0^{\pi/2}\sin^2\theta\cos^2\theta d\theta = a^4\int_0^{\pi/2}\frac{1}{2}(1-\cos 2\theta)\frac{1}{2}(1+\cos 2\theta)\,d\theta = \frac{a^4}{4}\int_0^{\pi/2}(1-\cos^2 2\theta)d\theta$$

$$= \frac{a^4}{4}\int_0^{\pi/2}\sin^2 2\theta d\theta = \frac{a^4}{4}\int_0^{\pi/2}\frac{1}{2}(1-\cos^2 4\theta)d\theta = \frac{a^4}{8}\left(\theta - \frac{1}{4}\sin 4\theta\right)\Big|_0^{\pi/2} = \frac{a^4\pi}{16}$$

□

6. $\displaystyle\int_0^{0.6}\frac{x^2}{\sqrt{9-25x^2}}dx$

Solution: Let's simplify the integral a little bit:

$$\int_0^{0.6}\frac{x^2}{\sqrt{9-25x^2}}dx = \int_0^{0.6}\frac{x^2}{\sqrt{9(1-\frac{25}{9}x^2)}}dx = \int_0^{0.6}\frac{x^2}{3\sqrt{1-(\frac{5}{3}x)^2}}dx$$

Let $\frac{5}{3}x = \sin\theta$, then $x = \frac{3}{5}\sin\theta$ and $dx = \frac{3}{5}\cos\theta d\theta$. We should also change the bounds. When $x = 0.6, \sin\theta = 1 \Rightarrow \theta = \frac{pi}{2}$. When $x = 0, \sin\theta = 0 \Rightarrow \theta = 0$. Then

$$\int_0^{0.6}\frac{x^2}{3\sqrt{1-(\frac{5}{3}x)^2}}dx = \int_0^{\pi/2}\frac{(\frac{3}{5}\sin\theta)^2}{3\sqrt{1-\sin^2\theta}}\cdot\frac{3}{5}\cos\theta d\theta = \int_0^{\pi/2}\frac{\left(\frac{3}{5}\right)^3\sin^2\theta\cos\theta}{3\cos\theta}d\theta$$

$$= \frac{9}{125}\int_0^{\pi/2} \sin^2\theta d\theta = \frac{9}{125}\cdot\frac{1}{2}\int_0^{\pi/2}(1-\cos 2\theta)d\theta = \frac{9}{250}\left(\theta - \frac{1}{2}\sin 2\theta\right)\Big|_0^{\pi/2} = \frac{9\pi}{500}$$

□

7. $\int_0^1 \sqrt{x^2+1}dx$

Solution: Let $x = \tan\theta$, then $dx = \sec^2\theta d\theta$. When $x = 1, tan\theta = 1 \Rightarrow \theta = \frac{\pi}{4}$. When $x = 0, \tan\theta = 0 \Rightarrow \theta = 0$. Then

$$\int_0^1 \sqrt{x^2+1}dx = \int_0^{\pi/4}\sqrt{\tan^2\theta+1}\cdot\sec^2\theta d\theta = \int_0^{\pi/4}\sec^3\theta d\theta$$

We did this integral on a previous practice sheet, you just need to use integration by parts:

$$\int_0^{\pi/4}\sec^3\theta d\theta = \frac{1}{2}(\sec\theta\tan\theta + \ln|\sec\theta+\tan\theta|)\Big|_0^{\pi/4} = \frac{1}{2}\left(\sqrt{2}+\ln(\sqrt{2}+1)\right)$$

□

8. $\displaystyle\int \frac{x}{\sqrt{x^2+x+1}}dx$

Solution: Complete the square: $x^2+x+1 = (x+\frac{1}{2})^2+\frac{3}{4}$. Then

$$\int\frac{x}{\sqrt{x^2+x+1}}dx = \int\frac{x}{\sqrt{(x+\frac{1}{2})^2+\frac{3}{4}}}dx$$

Let $u = x+\frac{1}{2}$, then $du = dx$:

$$\int\frac{x}{\sqrt{(x+\frac{1}{2})^2+\frac{3}{4}}}dx = \int\frac{u-\frac{1}{2}}{\sqrt{u^2+\frac{3}{4}}}du = \int\frac{u}{\sqrt{u^2+\frac{3}{4}}}du - \frac{1}{2}\int\frac{1}{\sqrt{u^2+\frac{3}{4}}}du$$

We will use the same substitution for both integrals. Let $u = \frac{\sqrt{3}}{2}\tan\theta$, then $du = \frac{\sqrt{3}}{2}\sec^2\theta d\theta$:

$$\int\frac{u}{\sqrt{u^2+\frac{3}{4}}}du - \frac{1}{2}\int\frac{1}{\sqrt{u^2+\frac{3}{4}}}du = \int\frac{\frac{\sqrt{3}}{2}\tan\theta}{\sqrt{\frac{3}{4}\tan^2\theta+\frac{3}{4}}}\frac{\sqrt{3}}{2}\sec^2\theta d\theta - \frac{1}{2}\int\frac{1}{\sqrt{\frac{3}{4}\tan^2\theta+\frac{3}{4}}}\frac{\sqrt{3}}{2}\sec^2\theta d\theta$$

$$= \int\frac{\frac{\sqrt{3}}{2}\tan\theta\frac{\sqrt{3}}{2}\sec^2\theta}{\frac{\sqrt{3}}{2}\sec\theta}d\theta - \frac{1}{2}\int\frac{\frac{\sqrt{3}}{2}\sec^2\theta}{\frac{\sqrt{3}}{2}\sec\theta}d\theta = \frac{\sqrt{3}}{2}\int\sec\theta\tan\theta d\theta - \frac{1}{2}\int\sec\theta d\theta$$

$$= \frac{\sqrt{3}}{2}\sec\theta - \frac{1}{2}\ln|\sec\theta+\tan\theta| + C$$

Now we need to plug u back in. Our substitution was $u = \frac{\sqrt{3}}{2}\tan\theta$, so our triangle with have u on the opposite side, $\frac{\sqrt{3}}{2}$ on the adjacent side, and $\sqrt{u^2+\frac{3}{4}}$ on the hypotenuse. So then

$$\frac{\sqrt{3}}{2}\sec\theta - \frac{1}{2}\ln|\sec\theta+\tan\theta| + C = \sqrt{u^2+\frac{3}{4}} - \frac{1}{2}\ln\left|\frac{2\sqrt{u^2+\frac{3}{4}}}{\sqrt{3}} + \frac{2u}{\sqrt{3}}\right| + C$$

Next plug back in x:

$$\int \frac{x}{\sqrt{(x+\frac{1}{2})^2+\frac{3}{4}}}dx = \sqrt{\left(x+\frac{1}{2}\right)^2+\frac{3}{4}} - \frac{1}{2}\ln\left|\frac{2\sqrt{(x+\frac{1}{2})^2+\frac{3}{4}}}{\sqrt{3}} + \frac{2(x+\frac{1}{2})}{\sqrt{3}}\right| + C$$

□

9. $\displaystyle\int \frac{x^2}{(3+4x-4x^2)^{3/2}}dx$

Solution: Complete the square: $3+4x-4x^2 = 3-4(x^2-x) = 3-4(x^2-x+\frac{1}{4}-\frac{1}{4}) = 3-4(x-\frac{1}{2})^2+1 = 4-4(x-\frac{1}{2})^2$. So then

$$\int \frac{x^2}{(3+4x-4x^2)^{3/2}}dx = \int \frac{x^2}{\left(\sqrt{4-4(x-\frac{1}{2})^2}\right)^3}dx = \int \frac{x^2}{8\left(\sqrt{1-(x-\frac{1}{2})^2}\right)^3}dx$$

Let $u = x-\frac{1}{2}$, $du = dx$:

$$\int \frac{x^2}{8\left(\sqrt{1-(x-\frac{1}{2})^2}\right)^3}dx = \int \frac{(u+\frac{1}{2})^2}{8\left(\sqrt{1-u^2}\right)^3}dx = \frac{1}{8}\int \frac{u^2+u+\frac{1}{4}}{(\sqrt{1-u^2})^3}du$$

$$= \frac{1}{8}\left(\int \frac{u^2}{(\sqrt{1-u^2})^3}du + \int \frac{u}{(\sqrt{1-u^2})^3}du + \frac{1}{4}\int \frac{1}{(\sqrt{1-u^2})^3}du\right)$$

Now let $u = \sin\theta$, so $du = \cos\theta d\theta$:

$$= \frac{1}{8}\left(\int \frac{\sin^2\theta}{(\sqrt{1-\sin^2\theta})^3}\cos\theta d\theta + \int \frac{\sin\theta}{(\sqrt{1-\sin^2\theta})^3}\cos\theta d\theta + \frac{1}{4}\int \frac{1}{(\sqrt{1-\sin^2\theta})^3}\cos\theta d\theta\right)$$

$$= \frac{1}{8}\left(\int \frac{\sin^2\theta\cos\theta}{\cos^3\theta}d\theta + \int \frac{\sin\theta\cos\theta}{\cos^3\theta}d\theta + \frac{1}{4}\int \frac{\cos\theta}{\cos^3\theta}d\theta\right)$$

$$= \frac{1}{8}\left(\int \frac{\sin^2\theta}{\cos^2\theta}d\theta + \int \frac{\sin\theta}{\cos^2\theta}d\theta + \frac{1}{4}\int \frac{1}{\cos^2\theta}d\theta\right)$$

$$= \frac{1}{8}\left(\int \tan^2\theta d\theta + \int \sec\theta\tan\theta d\theta + \frac{1}{4}\int \sec^2\theta d\theta\right)$$

$$= \frac{1}{8}\left(\int (\sec^2\theta - 1)d\theta + \int \sec\theta\tan\theta d\theta + \frac{1}{4}\int \sec^2\theta d\theta\right)$$

$$= \frac{1}{8}\left(\tan\theta - \theta + \sec\theta + \frac{1}{4}\tan\theta\right) + C = \frac{1}{8}\left(\frac{5}{4}\tan\theta - \theta + \sec\theta\right) + C$$

Now we need to plug u back in. Our substitution was $u = \sin\theta$, so the opposite side will be u, the hypotenuse will be 1, and the adjacent side will be $\sqrt{1-u^2}$:

$$\frac{1}{8}\left(\frac{5}{4}\tan\theta - \theta + \sec\theta\right) + C = \frac{1}{8}\left(\frac{5}{4}\frac{u}{\sqrt{1-u^2}} - \sin^{-1}u + \frac{1}{\sqrt{1-u^2}}\right) + C$$

Now plug x back in:

$$\int \frac{x^2}{(3+4x-4x^2)^{3/2}}dx = \frac{1}{8}\left(\frac{5}{4}\frac{x-\frac{1}{2}}{\sqrt{1-(x-\frac{1}{2})^2}} - \sin^{-1}\left(x-\frac{1}{2}\right) + \frac{1}{\sqrt{1-(x-\frac{1}{2})^2}}\right) + C$$

□

10. $\int x\sqrt{1-x^4}dx$

Solution:

$$\int x\sqrt{1-x^4}dx = \int x\sqrt{1-(x^2)^2}dx$$

Let $u = x^2$, then $du = 2xdx$:

$$\int x\sqrt{1-(x^2)^2}dx = \frac{1}{2}\int \sqrt{1-u^2}du$$

Now let $u = \sin\theta$, then $du = \cos\theta d\theta$:

$$\frac{1}{2}\int \sqrt{1-u^2}du = \frac{1}{2}\int \sqrt{1-\sin^2\theta}\cdot\cos\theta d\theta = \frac{1}{2}\int \cos^2\theta d\theta = \frac{1}{4}\int (1+\cos 2\theta)d\theta$$

$$= \frac{1}{4}\left(\theta + \frac{1}{2}\sin 2\theta\right) + C = \frac{1}{4}(\theta + \sin\theta\cos\theta) + C$$

Plug back in u. Since $u = \sin\theta$, the opposite side will be u, the hypotenuse will be 1, and the adjacent side will be $\sqrt{1-u^2}$:

$$\frac{1}{4}(\theta + \sin\theta\cos\theta) + C = \frac{1}{4}\left(\sin^{-1}u + u\sqrt{1-u^2}\right) + C$$

Then plug back in x:

$$\int x\sqrt{1-x^4}dx = \frac{1}{4}\left(\sin^{-1}(x^2) + x^2\sqrt{1-x^4}\right) + C$$

□

11. $\int \frac{dt}{\sqrt{t^2-6t+13}}$

Solution: Complete the square: $t^2-6t+13 = t^2-6t+9-9+13 = (t-3)^2+4$. Then

$$\int \frac{dt}{\sqrt{t^2-6t+13}} = \int \frac{dt}{\sqrt{(t-3)^2+4}}$$

Let $t-3 = 2\tan\theta$, then $dt = 2\sec^2\theta d\theta$:

$$\int \frac{1}{\sqrt{4\tan^2\theta+4}}\cdot 2\sec^2\theta d\theta = \int \frac{2\sec^2\theta}{2\sec\theta}d\theta = \int \sec\theta d\theta = \ln|\sec\theta+\tan\theta| + C$$

Our substitution was $t - 3 = 2\tan\theta$, so the opposite side will be $t - 3$, the adjacent side will be 2, and the hypotenuse will be $\sqrt{(t-3)^2 + 4}$:

$$\int \frac{dt}{\sqrt{t^2 - 6t + 13}} = \ln|\sec\theta + \tan\theta| + C = \ln\left|\frac{\sqrt{(t-3)^2+4}}{2} + \frac{t-3}{2}\right| + C$$