Practice Problems: Trig Integrals (Solutions)

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The following are solutions to the Trig Integrals practice problems posted on November 9.

1. $\int \sec x dx$

   Note: This is an integral you should just memorize so you don't need to repeat this process again.

   Solution:
$$\int \sec x dx = \int \sec x \frac{\sec x + \tan x}{\sec x + \tan x} dx = \int \frac{\sec^2 x + \sec x \tan x}{\sec x + \tan x} dx$$
   Let $w = \sec x + \tan x$, so $du = (\sec x \tan x + \sec^2 x)dx$:
$$\int \frac{\sec^2 x + \sec x \tan x}{\sec x + \tan x} dx = \int \frac{1}{w} dw = \ln |w| + C$$
   Plug back in w:
$$\int \sec x dx = \ln |\sec x + \tan x| + C$$
   □

2. $\int \sec^3 x dx$

   Solution: Rewrite:
$$\int \sec^3 x dx = \int \sec x \cdot \sec^2 x dx$$
   Use integration by parts. Let $u = \sec x, dv = \sec^2 x dx$. Then $du = \sec x \tan x dx$ and $v = \tan x$:
$$\int \sec^3 x dx = \sec x \tan x - \int \sec x \tan^2 x dx = \sec x \tan x - \int \sec x(\sec^2 x - 1)dx$$
$$= \sec x \tan x - \int \sec^3 x dx + \int \sec x dx = \sec x \tan x + \ln |\sec x + \tan x| - \int \sec^3 x dx$$
   Notice on the right side we have the same integral as what we started with, so move it over to the left side:
$$2 \int \sec^3 x dx = \sec x \tan x + \ln |\sec x + \tan x|$$
   Divide by 2 and add C:
$$\int \sec^3 x dx = \frac{1}{2} (\sec x \tan x + \ln |\sec x + \tan x|) + C$$
   □

3. $\int \cos^4 x dx$

Solution: Since we have an even power of cos, we need to use the half angle identity:

$$\int \cos^4 x dx = \int (\cos^2 x)^2 dx = \int \left(\frac{1}{2}(1+\cos 2x)\right)^2 dx = \frac{1}{4}\int (1 + 2\cos 2x + \cos^2 2x)dx$$

Use half angle again:

$$\frac{1}{4}\int \left(1 + 2\cos 2x + \frac{1}{2}(1+\cos 4x)\right) dx = \frac{1}{4}\int \left(\frac{3}{2} + 2\cos 2x + \frac{1}{2}\cos 4x\right) dx$$
$$= \frac{1}{4}\left(\frac{3}{2}x + \sin 2x + \frac{1}{8}\sin 4x\right) + C$$

□

4. $\int t\sin^2 t dt$

Solution: Use half angle identity:

$$\int t\sin^2 t dt = \int t\left(\frac{1}{2}(1-\cos 2t)\right) dt = \frac{1}{2}\left(\int t dt - \int t\cos 2t dt\right)$$

The first integral is straightforward, use integration by parts (tabular method) on the second with $u = t, dv = \cos 2t dt$:

$$\int t\sin^2 t dt = \frac{1}{2}\left(\frac{1}{2}t^2 - \frac{1}{2}t\sin 2t - \frac{1}{4}\cos 2t\right) + C$$

□

5. $\int \frac{\sin^3\sqrt{x}}{\sqrt{x}} dx$

Solution: Let $w = \sqrt{x}$, so $dw = \frac{1}{2\sqrt{x}}dx \Rightarrow 2dw = \frac{1}{\sqrt{x}}dx$:

$$\int \frac{\sin^3\sqrt{x}}{\sqrt{x}} dx = 2\int \sin^3 w dw = 2\int \sin w \cdot \sin^2 w dw = 2\int \sin w(1-\cos^2 w)dw$$

Let $y = \cos w$, so $dy = -\sin w dw$:

$$2\int \sin w(1-\cos^2 w)dw = -2\int (1-y^2)dy = -2\left(y - \frac{1}{3}y^3\right)$$

Plug back in w:

$$-2\left(y - \frac{1}{3}y^3\right) = -2\left(\cos w - \frac{1}{3}\cos^3 w\right)$$

Plug back in x and add C:

$$-2\left(\cos w - \frac{1}{3}\cos^3 w\right) = -2\left(\cos\sqrt{x} - \frac{1}{3}\cos^3\sqrt{x}\right) + C$$

□

6. $\int_0^\pi \sin^2 t \cos^4 t dt$

Solution: You can use half angle identity on this problem, but you would need to use it several times. I don't think you would see a problem like this on your exam, but it is nice to practice anyway.

$$\int_0^\pi \sin^2 t \cos^4 t dt = \int_0^\pi \sin^2 t \cos^2 t \cos^2 t dt = \int_0^\pi (\sin t \cos t)^2 \left(\frac{1}{2}(1 + \cos 2t)\right) dt$$

$$\frac{1}{2}\int_0^\pi \left(\frac{1}{2}\sin 2t\right)^2 (1 + \cos 2t)dt = \frac{1}{8}\int_0^\pi (\sin 2t)^2(1 + \cos 2t)dt$$

$$= \frac{1}{8}\left(\int_0^\pi (\sin 2t)^2 dt + \int_0^\pi (\sin 2t)^2 \cos 2t dt\right)$$

Let's look at these integrals separately. The left integral we need to use half angle identity:

$$\int_0^\pi \sin^2 2t dt = \frac{1}{2}\int_0^\pi (1 - \cos 2t)dt = \frac{1}{2}\left(t - \frac{1}{2}\sin 2t\right)\Big|_0^\pi = \frac{\pi}{2}$$

Now let's look at the right integral. Use the substitution $w = \sin 2t$, then $dw = 2\cos 2t dt$:

$$\int_0^\pi (\sin 2t)^2 \cos 2t dt = \frac{1}{2}\int_0^0 w^2 dw = 0$$

So the final answer is:

$$\int_0^\pi \sin^2 t \cos^4 t dt = \frac{1}{8}\left(\frac{\pi}{2}\right) = \frac{\pi}{16}$$

□

7. $\int_0^{\pi/2} (2 - \sin\theta)^2 d\theta$

Solution: Multiply this all out and use half angle identity:

$$\int_0^{\pi/2} (2 - \sin\theta)^2 d\theta = \int_0^{\pi/2} (4 - 4\sin\theta + \sin^2\theta)d\theta = \int_0^{\pi/2} \left(4 - 4\sin\theta + \frac{1}{2} - \frac{1}{2}\cos 2\theta\right) d\theta$$

$$= \int_0^{\pi/2} \left(\frac{9}{2} - 4\sin\theta - \frac{1}{2}\cos 2\theta\right) d\theta = \frac{9}{2}\theta + 4\cos\theta - \frac{1}{4}\sin 2\theta\Big|_0^{\pi/2} = \frac{9\pi}{4} - 4$$

□

8. $\int \cos^2 x \sin 2x dx$

Solution:

$$\int \cos^2 x \sin 2x dx = \int \cos^2 x \cdot 2\sin x \cos x dx = 2\int \sin x \cos^3 x dx$$

Let $w = \cos x, dw = -\sin x dx$:

$$2\int \sin x \cos^3 x dx = -2\int w^3 dw = -\frac{1}{2}w^4 + c = -\frac{1}{2}\cos^4 x + C$$

□

9. $\int \tan x \sec^3 x dx$

   Solution:

$$\int \tan x \sec^3 x dx = \int \sec^2 x \cdot \sec x \tan x dx$$

   Let $w = \sec x, dw = \sec x \tan x dx$:

$$\int \sec^2 x \cdot \sec x \tan x dx = \int w^2 dw = \frac{1}{3}w^3 + C = \frac{1}{3}\sec^3 x + C$$

   □

10. $\int x \sec x \tan x dx$

    Solution: Use integration by parts with $u = x$, $dv = \sec x \tan x dx$. Then $du = dx, v = \sec x$:

$$\int x \sec x \tan x dx = x \sec x - \int \sec x dx = x \sec x - \ln|\sec x + \tan x| + C$$

    □

11. $\int \csc x dx$

    Note: This is similar to the first problem. This is an integral you should just memorize so you don't need to repeat this process again.

    Solution:

$$\int \csc x dx = \int \csc x \frac{\csc x - \cot x}{\csc x - \cot x} dx = \int \frac{\csc^2 x - \csc x \cot x}{\csc x - \cot x} dx$$

    Let $w = \csc x - \cot x$. Then $dw = (-\csc x \cot x + \csc^2 x)dx$:

$$\int \frac{\csc^2 x - \csc x \cot x}{\csc x - \cot x} dx = \int \frac{1}{w} dw = \ln|w| + C = \ln|\csc x - \cot x| + C$$

    □

12. $\int \cot^3 x dx$

    Solution:

$$\int \cot^3 x dx = \int \cot x \cot^2 x dx = \int \cot x(\csc^2 x - 1)dx = \int \cot x \csc^2 x dx - \int \cot x dx$$

$$= \int \csc x \cdot \csc x \cot x dx - \int \cot x dx$$

Let's look at the first integral. Let $w = \csc x$, then $dw = -\csc x \cot x dx$:

$$\int \csc x \cdot \csc x \cot x dx = -\int w dw = -\frac{1}{2}w^2 = -\frac{1}{2}\csc^2 x$$

Now let's look at the second integral. Rewrite it and let $y = \sin x$ so $dy = \cos x dx$:

$$\int \cot x dx = \int \frac{\cos x}{\sin x} dx = \int \frac{1}{y} dy = \ln |y| = \ln |\sin x|$$

Now combine the two answers and add C:

$$\int \cot^3 x dx = -\frac{1}{2}\csc^2 x + \ln |\sin x| + C$$

□

13. $\int \sin 8x \cos 5x dx$

Solution: I don't think you would see a problem like this on your exam, but it is nice to practice anyway. There is a trig identity listed on page 476 of your text: $\sin A \cos B = \frac{1}{2}[\sin(A-B) - \sin(A+B)]$. You can also derive this equation yourself.

$$\int \sin 8x \cos 5x dx = \frac{1}{2}\left(\int (\sin 3x + \sin 13x)\, dx\right) = \frac{1}{2}\left(-\frac{1}{3}\cos 3x - \frac{1}{13}\cos 13x\right) + C$$

□

14. $\int \cos \pi x \cos 4\pi x dx$

Solution: Similar to the previous problem, I don't think you would see a problem like this on your exam. On page 476 of your text is the identity $\cos A \cos B = \frac{1}{2}[\cos(A-B) + \cos(A+B)]$.

$$\int \cos \pi x \cos 4\pi x dx = \frac{1}{2}\left(\int (\cos(-3\pi x) + \cos 5\pi x) dx\right) = \frac{1}{2}\left(\int (\cos 3\pi x + \cos 5\pi x) dx\right)$$

$$= \frac{1}{2}\left(\frac{1}{3\pi}\sin 3\pi x + \frac{1}{5\pi}\sin 5\pi x\right) + C$$

□

15. $\int_0^{\pi/6} \sqrt{1 + \cos 2x} dx$

Solution: This is using the half identity backwards.

$$\int_0^{\pi/6} \sqrt{1+\cos 2x} dx = \int_0^{\pi/6} \sqrt{2 \cdot \frac{1}{2}(1+\cos 2x)} dx = \int_0^{\pi/6} \sqrt{2\cos^2 x} dx = \sqrt{2}\int_0^{\pi/6} \cos x dx$$

$$= \sqrt{2}\sin x\Big|_0^{\pi/6} = \frac{\sqrt{2}}{2}$$

□

16. $\int_0^{\pi/4} \sqrt{1-\cos 4\theta} d\theta$

Solution: This is using the half identity backwards.

$$\int_0^{\pi/4} \sqrt{1-\cos 4\theta} d\theta = \int_0^{\pi/4} \sqrt{2 \cdot \frac{1}{2}(1-\cos 4\theta)} d\theta = \int_0^{\pi/4} \sqrt{2\sin^2 2\theta} d\theta = \sqrt{2}\int_0^{\pi/4} \sin 2\theta d\theta$$

$$= \sqrt{2}\left(-\frac{1}{2}\cos 2\theta\right)\Big|_0^{\pi/4} = \frac{\sqrt{2}}{2}$$

□

17. $\int \frac{1-\tan^2 x}{\sec^2 x} dx$

Solution:

$$\int \frac{1-\tan^2 x}{\sec^2 x} dx = \int (\cos^2 x - \sin^2 x) dx = \int \cos 2x dx = \frac{1}{2}\sin 2x + C$$

□

18. $\int \frac{dx}{\cos x - 1}$

Solution: Multiply by the conjugate:

$$\int \frac{dx}{\cos x - 1} = \int \frac{1}{\cos x - 1} \cdot \frac{\cos x + 1}{\cos x + 1} dx = \int \frac{\cos x + 1}{\cos^2 x - 1} dx = \int \frac{\cos x + 1}{-\sin^2 x} dx$$

$$= \int (-\csc x \cot x - \csc^2 x) dx = \csc x + \cot x + C$$

□

19. $\int x \tan^2 x dx$

Solution: Use the identity $\tan^2 x = \sec^2 x - 1$:

$$\int x \tan^2 x dx = \int x(\sec^2 x - 1) dx = \int x \sec^2 x dx - \int x dx$$

The last integral is no problemo. The first integral we need to use integration by parts. Let $u = x, dv = \sec^2 x$. Then $du = dx$, $v = \tan x$, so:

$$\int x \sec^2 x dx = x \tan x - \int \tan x dx$$

You can rewrite the last integral as $\int \frac{\sin x}{\cos x} dx$ and use the substitution $w = \cos x$. $\int \tan x dx = -\ln|\cos x|$, so:

$$\int x \sec^2 x dx = x \tan x + \ln|\cos x|$$

Plug that into the original integral:

$$\int x \tan^2 x dx = x \tan x + \ln|\cos x| - \frac{1}{2}x^2 + C$$

□

20. $\int x \sin^2 (x^2) dx$

Solution: Let $w = x^2$, $dw = 2xdx$:

$$\int x \sin^2 (x^2) dx = \frac{1}{2} \int \sin^2 w dw = \frac{1}{2} \cdot \frac{1}{2} \int (1 - \cos 2w) dw = \frac{1}{4}\left(w - \frac{1}{2}\sin 2w\right) + C$$

$$= \frac{1}{4}(w - \sin w \cos w) + C = \frac{1}{4}\left(x^2 - \sin(x^2)\cos(x^2)\right) + C$$

□

21. Find the area of the region bounded by the given curves: $y = \sin^3 x$, $y = \cos^3 x$, $\pi/4 \le x \le 5\pi/4$

Solution: To find the area we need to subtract the bottom function from the top function and then integrate over our domain:

$$A = \int_{\pi/4}^{5\pi/4} (\sin^3 x - \cos^3 x) dx = \int_{\pi/4}^{5\pi/4} \sin^3 x dx - \int_{\pi/4}^{5\pi/4} \cos^3 x dx$$

Let's look at the first integral:

$$\int_{\pi/4}^{5\pi/4} \sin^3 x dx = \int_{\pi/4}^{5\pi/4} \sin x \cdot \sin^2 x dx = \int_{\pi/4}^{5\pi/4} \sin x(1 - \cos^2 x) dx$$

Let $w = \cos x, dw = -\sin x dx$:

$$\int_{\pi/4}^{5\pi/4} \sin x(1 - \cos^2 x) dx = -\int_{\sqrt{2}/2}^{-\sqrt{2}/2} (1 - w^2) dw = \int_{-\sqrt{2}/2}^{\sqrt{2}/2} (1 - w^2) dw$$

$$= \left(w - \frac{1}{3}w^3\right)\Big|_{-\sqrt{2}/2}^{\sqrt{2}/2} = \sqrt{2} - \frac{\sqrt{2}}{6}$$

Now let's look at the second integral:

$$\int_{\pi/4}^{5\pi/4} \cos^3 x dx = \int_{\pi/4}^{5\pi/4} \cos x \cdot \cos^2 x dx = \int_{\pi/4}^{5\pi/4} \cos x(1-\sin^2 x)dx$$

Let $y = \sin x, dy = \cos x dx$:

$$\int_{\pi/4}^{5\pi/4} \cos x(1-\sin^2 x)dx = \int_{\sqrt{2}/2}^{-\sqrt{2}/2} (1-y^2)dy$$

$$= \left(y - \frac{1}{3}y^3\right)\Big|_{\sqrt{2}/2}^{-\sqrt{2}/2} = -\sqrt{2} + \frac{\sqrt{2}}{6}$$

Now plug this back in:

$$A = \int_{\pi/4}^{5\pi/4} \sin^3 x dx - \int_{\pi/4}^{5\pi/4} \cos^3 x dx = \left(\sqrt{2} - \frac{\sqrt{2}}{6}\right) - \left(-\sqrt{2} + \frac{\sqrt{2}}{6}\right) = 2\sqrt{2} - \frac{\sqrt{2}}{3} = \frac{5\sqrt{2}}{3}$$

□

22. Find the volume obtained by rotating the region bounded by the given curves about the specified axis: $y = \sec x, y = \cos x$, $0 \le x \le \pi/3$ about $y = -1$.

Solution: Use the washer method. The outer area is given by $A = \pi r^2 = \pi(\sec x + 1)^2$, and the inside area is given by $A = \pi r^2 = \pi(\cos x + 1)^2$:

$$V = \int_0^{\pi/3} (\text{outer area} - \text{inside area})dx = \pi \int_0^{\pi/3} ((\sec x + 1)^2 - (\cos x + 1)^2)dx$$

$$= \pi \int_0^{\pi/3} (\sec^2 x + 2\sec x - \cos^2 x - 2\cos x)dx = \pi \int_0^{\pi/3} \left(\sec^2 x + 2\sec x - \frac{1}{2} - \frac{1}{2}\cos 2x - 2\cos x\right) dx$$

$$= \pi \left(\tan x + 2\ln|\sec x + \tan x| - \frac{1}{2}x - \frac{1}{4}\sin 2x - 2\sin x\right)\Big|_0^{\pi/3}$$

$$= \pi \left(\tan\frac{\pi}{3} + 2\ln\left|\sec\frac{\pi}{3} + \tan\frac{\pi}{3}\right| - \frac{\pi}{6} - \frac{1}{4}\sin\frac{2\pi}{3} - 2\sin\frac{\pi}{3}\right) = \pi\left(2\ln(2+\sqrt{3}) - \frac{\pi}{6} - \frac{\sqrt{3}}{8}\right)$$

□