Recitation week 11

1) Use the method of partial fractions to evaluate the following integrals:

(a) $\int \frac{t^4-8}{t^3+2t^2}\,dt = I$ degree of numerator $= 4$

〃 〃 denominator $= 3$

$$\begin{array}{l|l} t^4-8 & t^3+2t^2 \\ \hline -(t^4+2t^3) & t-2 \\ -2t^3-8 & \\ -(-2t^3-4t^2) & \\ \hline 4t^2-8 & \end{array}$$

$$I = \int \left[t-2+\frac{4t^2-8}{t^3+2t^2} \right] dt$$

$$\frac{4t^2-8}{t^2(t+2)} = \underset{(t^2+2t)}{\frac{A}{t}} + \underset{(t+2)}{\frac{B}{t^2}} + \underset{(t^2)}{\frac{C}{t+2}}$$

$$A(t^2+2t)+B(t+2)+Ct^2 = 4t^2-8$$

$$t^2(A+c)+t(2A+B)+2B = 4t^2-8 \Rightarrow \left.\begin{array}{r} A+C=4 \\ 2A+B=0 \\ 2B=-8 \end{array}\right\} \begin{array}{l} A=2 \\ B=-4 \\ C=2 \end{array}$$

$$\therefore \frac{4t^2-8}{t^3+2t^2} = \frac{2}{t} + \frac{-4}{t^2} + \frac{2}{t+2}$$

So, $I = \int \left[t-2+2\cdot\frac{1}{t} - 4\cdot\frac{1}{t^2} + 2\cdot\frac{1}{t+2} \right] dt =$

$$= \frac{t^2}{2} - 2t + 2\ln|t| + 4\cdot\frac{1}{t} + 2\ln|t+2| + C_1, \quad C_1 = \text{constant}$$

(b) exc!

$$\int \frac{x^3+1}{x(x-1)^3}\,dx = \int \left[\frac{A}{x} + \frac{B}{x-1} + \frac{C}{(x-1)^2} + \frac{D}{(x-1)^3} \right] dx$$

Find A, B, C, D.

c) $\int \frac{x^3+3x}{x^4+2x^2+1}\,dx = I$

$$\frac{x^3+3x}{(x^2+1)^2} = \underset{(x^2+1)}{\frac{Ax+B}{x^2+1}} + \frac{Cx+D}{(x^2+1)^2}$$

$$(Ax+B)(x^2+1)+Cx+D = Ax^3+Bx^2+(A+C)x+(B+D) = x^3+3x$$

So, $\left.\begin{aligned} A&=1\\ B&=0\\ A+C&=3\\ B+D&=0 \end{aligned}\right\} \begin{aligned} A&=1\\ B&=0\\ C&=2\\ D&=0 \end{aligned}$

$C_1 \in \mathbb{R}$

$$I = \int\left(\frac{x}{x^2+1} + \frac{2x}{(x^2+1)^2}\right)dx = \frac{1}{2}\ln|x^2+1| - \frac{1}{x^2+1} + C_1$$

d) $\int \frac{2}{x^4-1}\,dx = I \qquad \frac{2}{(x-1)(x+1)(x^2+1)} = \frac{A}{x-1} + \frac{B}{x+1} + \frac{Cx+D}{x^2+1}$

$$A\underline{(x+1)(x^2+1)} + B\underline{(x-1)(x^2+1)} + \underline{(Cx+D)(x^2-1)} = 2$$

$$A(x^3+x^2+x+1) + B(x^3-x^2+x-1) + Cx^3+Dx^2-Cx-D = 2$$

$$x^3(A+B+C) + x^2(A-B+D) + x(A+B-C) + A-B-D = 2$$

$$\begin{aligned} A+B+C&=0\\ A-B+D&=0\\ A+B-C&=0\\ A-B-D&=2 \end{aligned} \qquad \begin{aligned} C&=0\\ A+B&=0\\ A-B&=1\\ D&=-1 \end{aligned} \qquad \begin{aligned} A&=\frac{1}{2}\\ B&=\frac{-1}{2} \end{aligned}$$

$$I = \int\left[\frac{1}{2}\cdot\frac{1}{x-1} - \frac{1}{2}\cdot\frac{1}{x+1} - \frac{1}{x^2+1}\right]dx = \frac{1}{2}\ln|x-1| - \frac{1}{2}\ln|x+1| - \arctan x + \text{Constant}$$

e) exc!

$$\int \frac{3x^2+2x+1}{(x+1)(x^2+x+1)}\,dx = \int\left[\frac{A}{x+1} + \frac{Bx+C}{x^2+x+1}\right]dx$$

Find A, B, C.

f) $\int_0^1 \frac{x}{x^3+1}\,dx$

$$\frac{x}{x^3+1} = \underset{(x^2-x+1)}{\frac{A}{x+1}} + \underset{(x+1)}{\frac{Bx+C}{x^2-x+1}}$$

$$x^2(A+B)+x(-A+B+C)+A+C = x$$

$A+B=0$

$-A+B+C=1$

$A+C=0$

$$\begin{array}{r} -A+B+C=1 \\ -\quad A+B\quad=0 \\ \hline -2A+C=1 \\ -\quad A+C=0 \\ \hline -3A=1 \end{array}$$

$A=-\frac{1}{3}$

$B=\frac{1}{3}$

$C=\frac{1}{3}$

$$\int_0^1 \frac{x}{x^3+1} = \int_0^1 \left[-\frac{1}{3}\,\frac{1}{x+1} + \frac{1}{3}\,\frac{x+1}{x^2-x+1}\right]dx = -\frac{1}{3}\ln|x+1|\Big|_{x=0}^{x=1} + \frac{1}{3}\int_0^1 \frac{x+1}{x^2-x+1}\,dx$$

$$\int_0^1 \frac{x+1}{x^2-x+1}\,dx = \int_0^1 \frac{x+1}{\left(x-\frac{1}{2}\right)^2+\frac{3}{4}}\,dx = \int_{-1/2}^{1/2} \frac{u+3/2}{u^2+\frac{3}{4}}\,du = \frac{4}{2}\int_{-1/2}^{1/2} \frac{2u+3}{4u^2+3}\,du =$$

$u = x-1/2$

$du = dx$

$x=0 \Rightarrow u=-1/2$

$x=1 \Rightarrow u=1/2$

$$= 2\left[\underbrace{\int_{-1/2}^{1/2} \frac{2u}{4u^2+3}\,du}_{0} + \int_{-1/2}^{1/2} \frac{1}{\left(\frac{2u}{\sqrt{3}}\right)^2+1}\,du\right] = 2\left[0+\frac{\sqrt{3}}{2}\arctan\left(\frac{2u}{\sqrt{3}}\right)\Big|_{u=-1/2}^{u=1/2}\right]$$

// odd func.

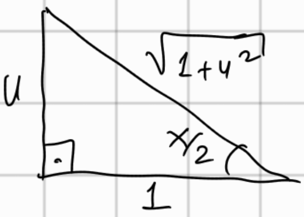
= exc!

(g) exc!

$$\int \frac{2-t^2}{t^3+3t^2+2t}\,dt$$

2) Use $u=\tan\left(\frac{x}{2}\right)$ substitution to evaluate $\int \frac{dx}{3\sin x + 4\cos x}$

Soln.



$$u = \tan\left(\frac{x}{2}\right)$$

$$du = \frac{1}{2}\left(1+\tan^2\left(\frac{x}{2}\right)\right)dx = \frac{1}{2}\left(1+u^2\right)dx$$

$$\therefore \quad dx = \frac{2}{1+u^2}\,du$$

$$\sin\left(2\cdot\frac{x}{2}\right) = 2\sin\left(\frac{x}{2}\right)\cos\left(\frac{x}{2}\right) = 2\cdot\frac{u}{\sqrt{1+u^2}}\cdot\frac{1}{\sqrt{1+u^2}} \Rightarrow \sin x = \frac{2u}{1+u^2}$$

$$\cos\left(2\cdot\frac{x}{2}\right) = \cos^2\left(\frac{x}{2}\right) - \sin^2\left(\frac{x}{2}\right) = \frac{1}{1+u^2} - \frac{u^2}{1+u^2} \Rightarrow \cos x = \frac{1-u^2}{1+u^2}$$

Hence,

$$\int\frac{dx}{3\sin x+4\cos x} = \int\frac{\frac{2}{1+u^2}\,du}{\frac{6u}{1+u^2}+\frac{4-4u^2}{1+u^2}} = \int\frac{1}{-2u^2+3u+2}\,du =$$

$$\begin{array}{cc} 2u & 1 \\ -u & 2 \end{array}$$

$$\int\frac{1}{(2u+1)(-u+2)}\,du = \int\left[\frac{A}{2u+1}+\frac{B}{-u+2}\right]du = I$$

$$A(-u+2)+B(2u+1)=1 \Rightarrow \begin{array}{l} -A+2B=0 \\ 2A+B=1 \end{array} \Rightarrow \begin{array}{l} B=1/5 \\ A=2/5 \end{array}$$

$$I = \int\left(\frac{2}{5}\cdot\frac{1}{2u+1}+\frac{1}{5}\,\frac{1}{-u+2}\right)du = \frac{\cancel{2}}{5}\cdot\frac{1}{\cancel{2}}\ln|2u+1| - \frac{1}{5}\ln|-u+2| + C$$

$$\left(u=\tan\left(\frac{x}{2}\right)\right) \quad I = \frac{1}{5}\ln\left|2\left(\tan\left(\frac{x}{2}\right)\right)+1\right| - \frac{1}{5}\ln\left|-\tan\left(\frac{x}{2}\right)+2\right| + C$$

3) Using inverse trigonometric substitution to evaluate the following integrals:

(a) $\int \frac{\sqrt{9-x^2}}{x}\,dx = I$

Recall: $\sqrt{a^2x^2-b^2}$, $x=\frac{b}{a}\sec\theta$

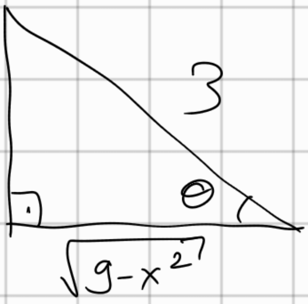
● $\sqrt{a^2-b^2x^2}$, $x=\frac{a}{b}\sin\theta$

$\sqrt{a^2+b^2x^2}$, $x=\frac{a}{b}\tan\theta$

Soln)

$x = 3\sin\theta$

$dx = 3\cos\theta\, d\theta$



$$I = \int \frac{\sqrt{9-9\sin^2\theta}}{3\sin\theta}\,3\cos\theta\, d\theta =$$

$$= \int \frac{\cancel{3}\sqrt{1-\sin^2\theta}\;3\cos\theta}{\cancel{3}\sin\theta}\,d\theta = \int \frac{3\cos\theta\cos\theta}{\sin\theta}\,d\theta =$$

$$= 3\left[\int \frac{1}{\sin\theta}\,d\theta - \int \sin\theta\, d\theta\right] = 3\left[\int \csc\theta\, d\theta + \cos\theta + C_1\right]$$

$$\int \csc\theta\, d\theta = \int \frac{\csc\theta(\csc\theta+\cot\theta)}{\csc\theta+\cot(\theta)}\,d\theta = \int \frac{\csc^2\theta+\csc\theta\cot\theta}{\csc\theta+\cot\theta}\,d\theta$$

$$= -\int \frac{-\csc^2\theta-\csc\theta\cot\theta}{\csc\theta+\cot\theta}\,d\theta = -\ln|\csc\theta+\cot\theta| + \underset{\text{Constant}}{C_2}$$

$u = \csc\theta + \cot\theta$

$du = (-\csc^2\theta - \csc\theta\cot\theta)\,d\theta$

∴

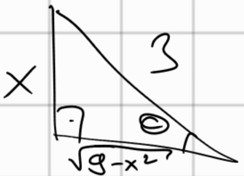
$$I = -3\ln|\csc\theta+\cot\theta| + \cos\theta + C =$$

$$= -3\ln\left|\frac{3}{x} + \frac{\sqrt{9-x^2}}{x}\right| + \frac{\sqrt{9-x^2}}{3} + C$$

$\csc\theta = \frac{1}{\sin\theta} = \frac{3}{x}$

$\cot(\theta) = \frac{\sqrt{9-x^2}}{x}$

$\cos\theta = \frac{\sqrt{9-x^2}}{3}$

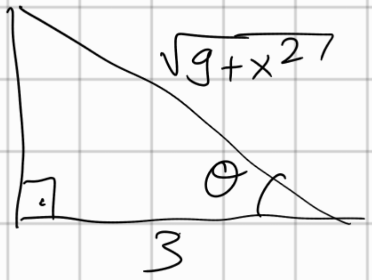


(b) $\int \frac{dx}{(9+x^2)^{3/2}} = I$

Soln)

$x = 3\tan\theta$

$dx = 3(1+\tan^2\theta)\,d\theta$



$$I = \int \frac{3(1+\tan^2\theta)}{(9+9\tan^2\theta)^{3/2}}\,d\theta = \int \frac{\cancel{3}\,\cancel{\sec^2\theta}}{\underset{3^2}{\cancel{3}^3}\,\sec^{\cancel{3}}\theta}\,d\theta = \int \frac{1}{9\sec\theta}\,d\theta$$

$1+\tan^2\theta = \sec^2\theta$

$9+9\tan^2\theta = 3^2\sec^2\theta = (3\sec\theta)^2$

$(9+9\tan^2\theta)^{3/2} = \left[(3\sec\theta)^{\cancel{2}}\right]^{3/\cancel{2}} =$

$= (3\sec\theta)^3 = 3^3\sec^3\theta$

$$= \frac{1}{9}\int \cos\theta\,d\theta = \frac{1}{9}\sin\theta + C =$$

$$= \frac{1}{9}\,\frac{x}{\sqrt{9+x^2}} + C$$

c) exc!

$$\int \frac{dx}{(1+x^2)^2}$$