Recitation Week 10

1) By properties of definite integral evaluate the followings:

(a) $\int_{-2}^{2}\left[\sinh(x)\ln\left(\sqrt{1+x^2}\right)+4x^5+7\right]dx$

exc. (b) $\int_{-1}^{2}\left[\sqrt{4-x^2}+|x+3|+|x-3|\right]dx$, exc. (c) $\int_{-1}^{4}\operatorname{sgn}(x)\,dx$

$$\operatorname{sgn}(x)=\begin{cases}-1 & \text{if } x<0\\ 0 & \text{if } x=0\\ 1 & \text{if } x>0\end{cases}$$

(a) $\sinh(x)=\dfrac{e^x-e^{-x}}{2}$

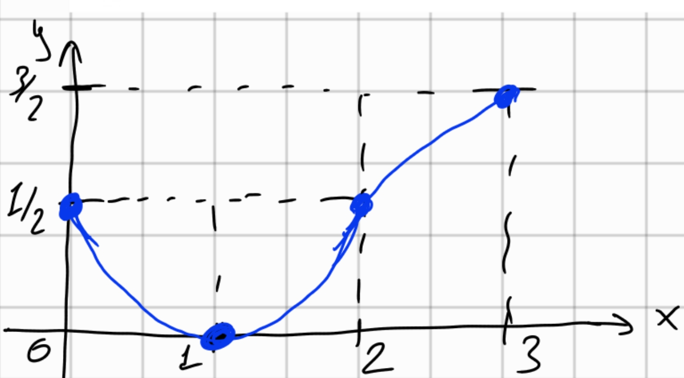
Since $\sinh(-x)=\dfrac{e^{-x}-e^{x}}{2}=-\left(\dfrac{e^x-e^{-x}}{2}\right)=-\sinh(x)$,

$\left.\begin{array}{l}\sinh(x) \text{ is an odd func.}\\ \ln\left(\sqrt{1+x^2}\right) \text{ is an even func.}\end{array}\right\}$ $\sinh(x)\cdot\ln\left(\sqrt{1+x^2}\right)$ is an odd func.

$$\int_{-2}^{2}\sinh(x)\ln\left(\sqrt{1+x^2}\right)dx=0$$

$$\underbrace{\int_{-2}^{2}(4x^5)\,dx}_{0 = \text{ odd func.}}+\int_{-2}^{2}\underbrace{7}_{\text{even func.}}dx=2\cdot\int_{0}^{2}7\,dx=2\cdot(7x)\Big|_{x=0}^{x=2}=\underline{\underline{28}}$$

2) Let $P=\{0,1,2,3\}$ be the partition of $[0,3]$ into three subintervals of equivalent length. Sketch roughly the graph of a non-negative continuous function $f:[0,3]\to\mathbb{R}$ such that the upper Riemann sum $U(f,P)=1/2+1/2+3/2$ and the lower Riemann sum $L(f,P)=0+0+1/2$.



$f(l_i)=0$

or

$\cancel{\Delta x=0}$

3) Express the limit $\lim_{n\to\infty} \sum_{i=1}^{n} \frac{2}{n}\left[1-\left(\frac{2i}{n}-1\right)^3\right]$ as a definite integral and evaluate it.

Soln)

$$\sum_{i=1}^{n} \underbrace{\frac{2}{n}}_{\Delta x} \underbrace{\left[1-\left(\frac{2i}{n}-1\right)^3\right]}_{f(x_i)}$$

$\Delta x = \frac{b-a}{n} = \frac{2}{n} \Rightarrow \boxed{b-a=2}$ //

Take $a=0$. Then $b=2$

Then $x_0 = 0 \Rightarrow x_i = 0 + i\cdot\Delta x = \frac{2i}{n}$

$$f(x_i) = 1-\left(\underbrace{\frac{2i}{n}}_{x_i}-1\right)^3 \Rightarrow f(x) = 1-(x-1)^3$$

$$\left(a=-1,\ b=1 \Rightarrow x_i = -1+\frac{2i}{n} \Rightarrow f(x) = 1-x^3\right)$$

Therefore,

$$\lim_{n\to\infty} \sum_{i=1}^{n} \frac{2}{n}\left[1-\left(\frac{2i}{n}-1\right)^3\right] = \int_0^2 \left[1-(x-1)^3\right]dx = \int_{-1}^{1}(1-x^3)\,dx =$$

$$= \left[x - \frac{(x-1)^4}{4}\right]\Bigg|_{x=0}^{x=2} = \left(2-\frac{1}{4}\right) - \left(0-\frac{(-1)^4}{4}\right) = \frac{7}{4}+\frac{1}{4} = 2 \ //$$

$$\left(\int_{-1}^{1}(1-x^3)\,dx = \left(x-\frac{x^4}{4}\right)\Bigg|_{x=-1}^{x=1} = \left(1-\frac{1}{4}\right) - \left(-1-\frac{1}{4}\right) = \frac{3}{4} - \left(-\frac{5}{4}\right) = 2 \ //\right)$$

4) Find the average value of the func. $f(x) = \frac{1}{x}$ over $\left[\frac{1}{3}, 3\right]$.

Soln)

Since f is continuous, f is integrable.

Hence,

$$\bar{f} = \frac{1}{3-\frac{1}{3}} \int_{1/3}^{3} \frac{1}{x}\,dx = \frac{3}{8} \ln|x| \Bigg|_{x=1/3}^{x=3} = \frac{3}{8}\ln(9)$$

$$= \frac{3}{8}\left(\ln(3) - \ln(1/3)\right)$$

5) Evaluate

$$\lim_{x\to 0} \frac{\displaystyle\int_{-2}^{0} \sin(t^2)\,dt - \int_{-2}^{x^2} \sin(t^2)\,dt}{\displaystyle\int_{\cos x}^{1} (e^{\sqrt{t}}-1)\,dt} = I$$

Soln)

Recall: (FTC)

$$\frac{d}{dx}\left(\int_{g(x)}^{h(x)} f(t)\,dt\right) = f(h(x))\cdot h'(x) - f(g(x))\cdot g'(x)$$

$$\frac{d}{dx}\left(\int_{-2}^{x^2} \sin(t^2)\,dt\right) = \sin(x^4)\cdot 2x - \sin(4)\cdot 0 = \sin(x^4)\cdot 2x$$

$$\frac{d}{dx}\left(\int_{\cos x}^{1} e^{\sqrt{t}-1}\,dt\right) = \cancel{-}\left(e^{\sqrt{\cos x}-1}\right)\cdot(\cancel{-}\sin x) = \sin x\cdot e^{\sqrt{\cos x}-1}$$

$$I \underset{\left[\frac{0}{0}\right]}{\overset{[L'H]}{=\!=}} \lim_{x\to 0} \frac{-2x\sin(x^4)\; x^4}{\sin x\cdot x^4\cdot e^{\sqrt{\cos x}-1}} = 0$$

$x^4 \to 0$ as $x\to 0$

$\frac{x}{\sin x} \to 1$ as $x\to 0$; $\frac{\sin(x^4)}{x^4}\to 1$: $t=x^4$, as $t\to 0$; $e^{\sqrt{\cos x}-1}\to 1$ as $x\to 0$

6) Evaluate the following definite or indefinite integrals:

(a) $\displaystyle\int_0^2 x^5\sqrt{x^2+3}\,dx = I$

Soln)

$u^2 = x^2+3 \Rightarrow \cancel{2}u\,du = \cancel{2}x\,dx$ So, $u\,du = x\,dx$

$x=0 \Rightarrow u=\sqrt{3}$, $x=2 \Rightarrow u=\sqrt{7}$

$$I = \int_{x=0}^{x=2} x^4\sqrt{x^2+3}\,x\,dx = \int_{x=0}^{x=2} (u^2-3)^2\sqrt{u^2}\,u\,du = \int_{\sqrt{3}}^{\sqrt{7}} [u^4-6u^2+9]\,u^2\,du =$$

$$= \left[\frac{u^7}{7} - 6\frac{u^5}{5} + \frac{9u^3}{3}\right]\Bigg|_{u=\sqrt{3}}^{u=\sqrt{7}} = \text{exc!}$$

(b) $\int \frac{2x+3}{4x^2+4x+3}\,dx = I$

Soln)

$$I = \int \frac{(2x+1+2)\,dx}{(2x+1)^2+2} = \underbrace{\int \frac{2x+1}{(2x+1)^2+2}\,dx}_{I_1} + \underbrace{\int \frac{2}{(2x+1)^2+2}\,dx}_{I_2}$$

For I_1:

$u = (2x+1)^2 + 2$

$du = 2(2x+1)\cdot 2\,dx$

$\frac{du}{4} = (2x+1)\,dx$

$$I_1 = \int \frac{1}{4}\cdot\frac{1}{u}\cdot du = \frac{1}{4}\ln|u| + C_1 \qquad C_1 \in \mathbb{R}$$

$$= \frac{1}{4}\ln\left|(2x+1)^2+2\right| + C_1$$

For I_2:

$$\int \frac{2}{(2x+1)^2+2}\,dx = \int \frac{\cancel{2}\cdot 1}{\cancel{2}\cdot\left[1+\left(\frac{2x+1}{\sqrt{2}}\right)^2\right]}\,dx = \int \frac{1}{1+\left(\frac{2x+1}{\sqrt{2}}\right)^2}\,dx = I_2$$

$u = \frac{2x+1}{\sqrt{2}}$ $\quad \frac{du}{\sqrt{2}} = dx$ $\quad$ So, $I_2 = \frac{1}{\sqrt{2}}\int \frac{1}{1+u^2}\cdot du = \frac{1}{\sqrt{2}}\arctan(u) + C_2$

$C_2 \in \mathbb{R}$

So, $I_2 = \frac{1}{\sqrt{2}}\arctan\left(\frac{2x+1}{\sqrt{2}}\right) + C_2$

So

$$I = \frac{1}{4}\ln\left|(2x+1)^2+2\right| + \frac{1}{\sqrt{2}}\arctan\left(\frac{2x+1}{\sqrt{2}}\right) + C, \quad C \in \mathbb{R}$$

constant

c) $\int \frac{\cos(\ln x)}{x^3}\,dx = I$

Soln)

$t = \ln(x) \Rightarrow dt = \frac{1}{x}\,dx \quad , \quad e^t = x$

$$I = \int \frac{\cos(t)}{e^{2t}}\cdot dt = \boxed{\int e^{-2t}\cos(t)\,dt = I}$$

By integration by parts:

Let $u = \cos(t)$ $\quad dv = e^{-2t}\,dt$ $\quad \left(\int u\,dv = uv - \int v\,du\right)$

$du = -\sin(t)\,dt$ $\quad v = \frac{e^{-2t}}{-2}$

So

$$I = \left[\frac{e^{-2t}\cos t}{-2} - \int \frac{e^{-2t}}{\cancel{-}2}\cdot(\cancel{-}\sin t)\,dt\right] = \left[\frac{e^{-2t}\cos t}{-2} - \frac{1}{2}\int e^{-2t}\sin t\,dt\right]$$

$$K = \int e^{-2t}\sin t\,dt = \frac{e^{-2t}\sin t}{-2} \cancel{-} \int \frac{e^{-2t}\cos t}{\cancel{-}2}\,dt$$

$u = \sin t,$ $\quad dv = e^{-2t}\,dt$

$du = \cos t\,dt$ $\quad v = \frac{e^{-2t}}{-2}\,dt$

$$K = \frac{1}{2}\left(-e^{-2t}\sin t + \int e^{-2t}\cos t\,dt\right)$$

Hence,

$$I = \frac{e^{-2t}\cos(t)}{-2} - \frac{1}{2}\cdot\frac{1}{2}\left(-e^{-2t}\sin t + \underbrace{\int e^{-2t}\cos(t)\,dt}_{I}\right)$$

So, $I = \frac{e^{-2t}\cos(t)}{-2} + \frac{1}{4}e^{-2t}\sin t - \frac{1}{4}I$

$$\therefore I = \frac{4}{5}\left(\frac{e^{-2t}\cos(t)}{-2} + \frac{e^{-2t}\sin t}{4}\right) + C = \frac{4}{5}\left(\frac{x^{-2}\cos(\ln(x))}{-2} + \frac{x^{-2}\sin(\ln(x))}{4}\right) + C$$

$t = \ln(x)$

$e^t = x$

$(C = \text{constant})$

$$= \frac{\sin(\ln(x)) - 2\cos(\ln(x))}{5x^2} + C$$

d) $\int \frac{\sin(2x)}{1+\sin^2(x)}\,dx = I$

Soln

$u = 1 + \sin^2 x$

$du = 2\sin x\cos x\,dx = \sin(2x)\,dx \Rightarrow I = \int \frac{1}{u}\,du = \ln|u| + C$

$= \ln|1+\sin^2(x)| + C$

e) $\int_1^e \sqrt[3]{x} \ln(x)\,dx = I$

Soln|

Let $u = \ln(x)$ and $\sqrt[3]{x}\,dx = dv$. Then $du = \frac{1}{x}dx$ and $v = \frac{3}{4}x^{4/3}$

So,

$$I = \left[\frac{3}{4}x^{4/3}\ln(x) - \int \frac{3}{4}\underbrace{x^{4/3}\cdot\frac{1}{x}}_{x^{1/3}}dx\right]\Bigg|_{x=1}^{x=e} = \left[\frac{3}{4}x^{4/3} - \frac{3}{4}\cdot\frac{3}{4}x^{4/3}\right]\Bigg|_{x=1}^{x=e} =$$

$$= \frac{3}{16}x^{4/3}\Bigg|_{x=1}^{x=e} = \frac{3}{16}\left(e^{4/3} - 1\right)$$

f) $\int x^2 \arcsin(x)\,dx = I$

Soln|

Let $u = \arcsin x$ and $dv = x^2 dx$. Then $du = \frac{1}{\sqrt{1-x^2}}dx$ & $v = \frac{x^3}{3}$.

So,

$$I = \frac{x^3}{3}\arcsin x - \int \frac{x^3}{3\sqrt{1-x^2}}dx = \frac{x^3}{3}\arcsin x - \frac{1}{3}\int \frac{x^3}{\sqrt{1-x^2}}dx$$

Since $\int \frac{x^3}{\sqrt{1-x^2}}dx = \int \frac{-(1-t^2)\,t\,dt}{t} = -t + \frac{t^3}{3} + C =$

Let $1-x^2 = t^2$

$-2x\,dx = 2t\,dt$

$$= -\sqrt{1-x^2} + \frac{(1-x^2)^{3/2}}{3} + C_1$$

$$I = \frac{x^3}{3}\arcsin x - \frac{1}{3}\left[-\sqrt{1-x^2} + \frac{(1-x^2)^{3/2}}{3}\right] + C, \quad C \in \mathbb{R}$$

exc!

g) $\int_2^{\sqrt{5}} x^2 e^{3x}\,dx$

h) $\int \sin^3 x \sec^5 x \, dx = I$

Soln.)

$$I = \int \sin^3 x \cdot \frac{1}{\cos^5 x} \, dx = \int \frac{\sin^3 x}{\cos^3 x} \cdot \frac{1}{\cos^2 x} \, dx = \int \tan^3 x \cdot \sec^2 x \, dx$$

Let $u = \tan x$. Then $du = \sec^2 x \, dx$. Hence,

$$I = \int u^3 \, du = \frac{u^4}{4} + C = \frac{\tan^4 x}{4} + C \quad , C \in \mathbb{R}$$

j) $\int \frac{\sec(\sqrt{x}) \tan(\sqrt{x})}{\sqrt{x}} \, dx = I$

Soln.)

Let $\sqrt{x} = u$. Then $\frac{1}{2\sqrt{x}} dx = du \Rightarrow \frac{1}{\sqrt{x}} dx = 2 \, du$.

So,

$$I = \int \sec(u) \tan(u) \cdot 2 \cdot du = 2 \sec(u) + C = 2 \sec(\sqrt{x}) + C,$$

$C \in \mathbb{R}$