Practice Problems: Integration of Rational Functions

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Solutions to the practice problems posted on November 30.

1. $\displaystyle\int \frac{5x+7}{x^3+2x^2-x-2}\,dx$

   Solution: From #2 on the Partial Fractions practice sheet, we know

$$\frac{5x+7}{x^3+2x^2-x-2} = \frac{2}{x-1} - \frac{1}{x+1} - \frac{1}{x+2}$$

   Then

$$\int \frac{5x+7}{x^3+2x^2-x-2}\,dx = \int \left(\frac{2}{x-1} - \frac{1}{x+1} - \frac{1}{x+2}\right) dx = 2\ln|x-1| - \ln|x+1| - \ln|x+2| + C$$

   □

2. $\displaystyle\int \frac{x^2+1}{x(x-1)^3}\,dx$

   Solution: From #3 on the Partial Fractions practice sheet, we know

$$\frac{x^2+1}{x(x-1)^3} = \frac{-1}{x} + \frac{1}{x-1} + \frac{2}{(x-1)^3}$$

   Then

$$\int \frac{x^2+1}{x(x-1)^3}\,dx = \int \left(\frac{-1}{x} + \frac{1}{x-1} + \frac{2}{(x-1)^3}\right) dx = -\ln|x| + \ln|x-1| - \frac{1}{(x-1)^2} + C$$

   □

3. $\displaystyle\int \frac{2x^2-x+4}{x^3+4x}\,dx$

   Solution: From #4 on the Partial Fractions practice sheet, we know

$$\frac{2x^2-x+4}{x(x^2+4)} = \frac{1}{x} + \frac{x-1}{x^2+4}$$

   Then

$$\int \frac{2x^2-x+4}{x(x^2+4)}\,dx = \int \left(\frac{1}{x} + \frac{x-1}{x^2+4}\right) dx = \int \left(\frac{1}{x} + \frac{x}{x^2+4} - \frac{1}{x^2+4}\right) dx$$

$$= \ln|x| + \frac{1}{2}\ln(x^2+4) - \frac{1}{2}\tan^{-1}\left(\frac{x}{2}\right) + C$$

   □

4. $\displaystyle\int \frac{x-4}{(2x-5)^2}\ dx$

Solution: From #9 on the Partial Fractions practice sheet, we know

$$\frac{x-4}{(2x-5)^2} = \frac{1/2}{2x-5} - \frac{3/2}{(2x-5)^2}$$

Then

$$\int \frac{x-4}{(2x-5)^2}\ dx = \int \left(\frac{1/2}{2x-5} - \frac{3/2}{(2x-5)^2}\right)\ dx$$

Let $u = 2x-5$, then $du = 2dx$:

$$\frac{1}{2}\int \left(\frac{1/2}{u} - \frac{3/2}{u^2}\right) du = \frac{1}{2}\left(\frac{1}{2}\ln|u| + \frac{3}{2u}\right) + C$$

Plug u back in:

$$= \frac{1}{2}\left(\frac{1}{2}\ln|2x-5| + \frac{3}{2(2x-5)}\right) + C$$

□

5. $\displaystyle\int \frac{x^4+x^3+x^2-x+1}{x(x^2+1)^2}\ dx$ *Solution:* From #13 on the Partial Fractions practice sheet, we know

$$\frac{x^4+x^3+x^2-x+1}{x(x^2+1)^2} = \frac{1}{x} + \frac{1}{x^2+1} + \frac{-x-2}{(x^2+1)^2}$$

Then

$$\int \frac{x^4+x^3+x^2-x+1}{x(x^2+1)^2}\ dx = \int \left(\frac{1}{x} + \frac{1}{x^2+1} + \frac{-x-2}{(x^2+1)^2}\right)\ dx$$
$$= \int \left(\frac{1}{x} + \frac{1}{x^2+1} - \frac{x}{(x^2+1)^2} - \frac{2}{(x^2+1)^2}\right)\ dx$$

The first two integrals are straightforward, so let's look at the last two.

For $\int \frac{x}{(x^2+1)^2}\ dx$, let $u = x^2+1$, then $du = 2xdx$, so

$$\int \frac{x}{(x^2+1)^2}\ dx = \frac{1}{2}\int \frac{1}{u^2}du = -\frac{1}{2u} = -\frac{1}{2(x^2+1)}$$

For $\int \frac{2}{(x^2+1)^2}\ dx$, let $x = \tan\theta$, then $dx = \sec^2\theta d\theta$:

$$\int \frac{2}{(x^2+1)^2}\ dx = \int \frac{2}{(\tan^2\theta+1)^2}\cdot\sec^2\theta d\theta = \int \frac{2}{\sec^2\theta}\ d\theta = 2\int \cos^2\theta d\theta = 2\cdot\frac{1}{2}\int (1+\cos 2\theta)d\theta$$
$$= \theta + \frac{1}{2}\sin 2\theta = \theta + \frac{1}{2}\cdot 2\sin\theta\cos\theta = \theta + \sin\theta\cos\theta = \tan^{-1}x + \frac{x}{x^2+1} + C$$

Plug these in and we get:

$$\ln|x| + \tan^{-1}x + \frac{1}{2(x^2+1)} - \tan^{-1}x - \frac{x}{x^2+1} + C = \ln|x| + \frac{1}{2(x^2+1)} - \frac{x}{x^2+1} + C$$

□