Practice Problems: Integration by Parts (Solutions)

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The following are solutions to the Integration by Parts practice problems posted November 9.

1. $\int e^x \sin x dx$

   Solution: Let $u = \sin x$, $dv = e^x dx$. Then $du = \cos x dx$ and $v = e^x$. Then

   $$\int e^x \sin x dx = e^x \sin x - \int e^x \cos x dx$$

   Now we need to use integration by parts on the second integral. Let $u = \cos x$, $dv = e^x dx$. Then $du = -\sin x dx$ and $v = e^x$. Then

   $$\int e^x \sin x dx = e^x \sin x - e^x \cos x - \int e^x \sin x dx$$

   The right integral is the same as the one we started with! Move it over:

   $$2 \int e^x \sin x dx = e^x \sin x - e^x \cos x$$

   And divide by 2:

   $$\int e^x \sin x dx = \frac{1}{2} (e^x \sin x - e^x \cos x)$$

   This is our final solution, so make sure to add your constant C:

   $$\int e^x \sin x dx = \frac{1}{2} (e^x \sin x - e^x \cos x) + C$$

   □

2. $\int (\sin^{-1} x)^2 dx$

   Solution: Let $u = (\sin^{-1} x)^2$, $dv = dx$. Then $du = \frac{2 \sin^{-1} x}{\sqrt{1-x^2}} dx$, $v = x$. Then

   $$\int (\sin^{-1} x)^2 dx = x(\sin^{-1} x)^2 - \int \frac{2x \sin^{-1} x}{\sqrt{1-x^2}} dx$$

   We need to use a substitution on the last integral. Let $w = \sin^{-1} x$. Then $dw = \frac{1}{\sqrt{1-x^2}} dx$ and $x = \sin w$. Just looking at the last integral, we have:

   $$\int \frac{2x \sin^{-1} x}{\sqrt{1-x^2}} dx = \int 2w \sin w dw$$

We can use integration by parts on this last integral by letting $u = 2w$ and $dv = \sin w dw$. Tabular method makes it rather quick:

$$\int 2w \sin w dw = 2w \cos w + 2 \sin w$$

At this point you can plug back in w:

$$\int 2w \sin w dw = 2 \sin^{-1} x \cos(\sin^{-1} x) + 2 \sin(\sin^{-1} x)$$

OR you can look at the triangle formed by our substitution for w. Since $x = \sin w$ then the hypotenuse will be 1, the opposite side will be x and the adjacent side will be $\sqrt{1-x^2}$. Then

$$\int 2w \sin w dw = 2\sqrt{1-x^2} \sin^{-1} x + 2x$$

Either of these solutions is fine. So then our integral will look like either one of the solutions below:

$$\int (\sin^{-1} x)^2 dx = x(\sin^{-1} x)^2 - (2 \sin^{-1} x \cos(\sin^{-1} x) + 2 \sin(\sin^{-1} x)) + C$$

$$\int (\sin^{-1} x)^2 dx = x(\sin^{-1} x)^2 - (2\sqrt{1-x^2} \sin^{-1} x + 2x) + C$$

□

3. $\int x \tan^2 x dx$

Solution: Use the identity $\tan^2 x = \sec^2 x - 1$:

$$\int x \tan^2 x dx = \int x(\sec^2 x - 1) dx = \int x \sec^2 x dx - \int x dx$$

The last integral is no problemo. The first integral we need to use integration by parts. Let $u = x, dv = \sec^2 x$. Then $du = dx$, $v = \tan x$, so:

$$\int x \sec^2 x dx = x \tan x - \int \tan x dx$$

You can rewrite the last integral as $\int \frac{\sin x}{\cos x} dx$ and use the substitution $w = \cos x$. $\int \tan x dx = -\ln|\cos x|$, so:

$$\int x \sec^2 x dx = x \tan x + \ln|\cos x|$$

Plug that into the original integral:

$$\int x \tan^2 x dx = x \tan x + \ln|\cos x| - \frac{1}{2}x^2 + C$$

□

4. $\int_0^1 t \cosh t dt$

Solution: This is quick with tabular method. Let $u = t$, $dv = \cosh t$:

$$\int_0^1 t \cosh t dt = t \sinh t - \cosh t \Big|_0^1 = \sinh(1) - \cosh(1) + \cosh(0)$$

You can leave your answer like this. If you want to evaluate it further, remember that $\sinh x = \dfrac{e^x - e^{-x}}{2}$ and $\cosh x = \dfrac{e^x + e^{-x}}{2}$. Then we see that $\sinh(1) = \frac{1}{2}(e^1 - e^{-1}), \cosh(1) = \frac{1}{2}(e^1 + e^{-1})$, and $\cosh 0 = 1$. Then

$$\int_0^1 t \cosh t dt = \sinh(1) - \cosh(1) + \cosh(0) = 1 - \frac{1}{e}$$

□

5. $\int z^3 e^z dx$

Solution: Tabular is the way to go with this baby. Let $u = z^3, dv = e^z dz$. Then

$$\int z^3 e^z dx = z^3 e^z - 3z^2 e^z + 6ze^z - 6e^z + C = e^z(z^3 - 3z^2 + 6z - 6) + C$$

□

6. $\int_1^{\sqrt{3}} \arctan(1/x) dx$

Solution: Let $u = \arctan(1/x), dv = dx$. Then $du = \frac{-dx}{x^2+1}$ (using chain rule), $v = x$:

$$\int_1^{\sqrt{3}} \arctan(1/x) dx = x \arctan(1/x) \Big|_1^{\sqrt{3}} + \int_1^{\sqrt{3}} \frac{x}{x^2+1} dx$$

The last integral you can use the substitution $w = x^2 + 1$. Then:

$$\int_1^{\sqrt{3}} \arctan(1/x) dx = x \arctan(1/x) + \frac{1}{2} \ln\left(x^2 + 1\right) \Big|_1^{\sqrt{3}}$$

$$= \sqrt{3} \arctan\left(\sqrt{3}\right) + \frac{1}{2} \ln 4 - \arctan\left(1\right) + \frac{1}{2} \ln 2 = \frac{\sqrt{3}\pi}{3} + \frac{1}{2} \ln 2 - \frac{\pi}{4}$$

□

7. $\int \cos x \ln(\sin x) dx$

Solution: We first need to do a substitution. Let $w = \sin x$, then $dw = \cos x dx$:

$$\int \cos x \ln(\sin x) dx = \int \ln w dw$$

Next use integration by parts with $u = \ln w, dv = dw$. Then $du = \frac{1}{w}dw, v = w$:

$$\int \ln w dw = w \ln w - \int dw = w \ln w - w$$

We need to plug back in w:

$$\int \cos x \ln(\sin x) dx = \sin x \ln(\sin x) - \sin x + C$$

□

8. $\int_1^2 \frac{(\ln x)^2}{x^3} dx$

You can do this problem a couple different ways. I will show you two solutions.

Solution I: First do the substitution $w = \ln x$. Then $dw = \frac{1}{x}dx$ and $x = e^w$. Then

$$\int_1^2 \frac{(\ln x)^2}{x^3} dx = \int_0^{\ln 2} \frac{w^2}{e^{2w}} dw = \int_0^{\ln 2} w^2 e^{-2w} dw$$

Tabular is easy on this guy:

$$\int_0^{\ln 2} w^2 e^{-2w} dw = -\frac{w^2}{2}e^{-2w} - \frac{w}{2}e^{-2w} - \frac{1}{4}e^{-2w}\Big|_0^{\ln 2} = -\frac{e^{-2w}}{2}\left(w^2 + w + \frac{1}{2}\right)\Big|_0^{\ln 2}$$

$$= -\frac{1}{8}\left((\ln 2)^2 + \ln 2 + \frac{3}{2}\right)$$

Solution II: Start of with integration by parts. Let $u = (\ln x)^2, dv = \frac{1}{x^3}dx$. Then $du = \frac{2\ln x}{x}dx, v = -\frac{1}{2x^2}$:

$$\int_1^2 \frac{(\ln x)^2}{x^3} dx = -\frac{(\ln x)^2}{2x^2}\Big|_1^2 + \int_1^2 \frac{\ln x}{x^3} dx$$

Do integration by parts again. Let $u = \ln x, dv = \frac{1}{x^3}dx$. Then $du = \frac{1}{x}dx, v = -\frac{1}{2x^2}$:

$$\int \frac{\ln x}{x^3} dx = -\frac{\ln x}{2x^2}\Big|_1^2 + \int_1^2 \frac{1}{2x^3} dx = \left(-\frac{\ln x}{2x^2} - \frac{1}{4x^2}\right)\Big|_1^2$$

Plugging this into the original integral we get:

$$\int_1^2 \frac{(\ln x)^2}{x^3} dx = \left(-\frac{(\ln x)^2}{2x^2} - \frac{\ln x}{2x^2} - \frac{1}{4x^2}\right)\Big|_1^2 = -\frac{1}{2x^2}\left((\ln x)^2 + \ln x + \frac{1}{2}\right)\Big|_1^2$$

$$= -\frac{1}{8}\left((\ln 2)^2 + \ln 2 + \frac{3}{2}\right)$$

□

9. $\int \cos\sqrt{x}dx$

Solution: First do the substitution $w = \sqrt{x}$. Then $dw = \dfrac{1}{2\sqrt{x}}dx \Rightarrow 2\sqrt{x}dw = dx \Rightarrow 2wdw = dx$:

$$\int \cos\sqrt{x}dx = \int 2w\cos wdw$$

Using tabular with $u = 2w, dv = \cos wdw$ we get:

$$\int 2w\cos wdw = 2w\sin w + 2\cos w + C$$

Plug back in w to get the final solution:

$$\int \cos\sqrt{x}dx = 2\sqrt{x}\sin\sqrt{x} + 2\cos\sqrt{x} + C$$

□

10. $\int_{\sqrt{\pi/2}}^{\sqrt{\pi}} \theta^3\cos(\theta^2)d\theta$

Note: There was a typo on the original, it should be $d\theta$ instead of dx.

Solution: Rewrite: $\int_{\sqrt{\pi/2}}^{\sqrt{\pi}} \theta^3\cos(\theta^2)d\theta = \int_{\sqrt{\pi/2}}^{\sqrt{\pi}} \theta\cdot\theta^2\cos(\theta^2)d\theta$. Then use the substitution $w = \theta^2$, so we have $dw = 2\theta d\theta$:

$$\int_{\sqrt{\pi/2}}^{\sqrt{\pi}} \theta^3\cos(\theta^2)d\theta = \frac{1}{2}\int_{\pi/2}^{\pi} w\cos wdw$$

Tabular makes this easy with $u = w$, $dv = \cos wdw$:

$$\frac{1}{2}\int_{\pi/2}^{\pi} w\cos wdw = \frac{1}{2}\left(w\sin w + \cos w\right)\Big|_{\pi/2}^{\pi} = -\frac{1}{2} - \frac{\pi}{4}$$

□

11. $\int x\ln(1+x)dx$

Solution: Use the substitution $w = 1 + x$. Then $dw = dx$ and $x = w - 1$:

$$\int x\ln(1+x)dx = \int (w-1)\ln wdw$$

Next use integration by parts with $u = \ln w, dv = (w-1)dw$. Then $du = \dfrac{1}{w}dw$ and $v = \left(\dfrac{1}{2}w^2 - w\right)$:

$$\int (w-1)\ln wdw = \left(\frac{1}{2}w^2 - w\right)\ln w - \int\left(\frac{1}{2}w - 1\right)dw$$

The right integral is straightforward, so

$$\int (w-1)\ln w dw = \left(\frac{1}{2}w^2 - w\right)\ln w - \frac{1}{4}w^2 + w + C$$

Next, plug back in w:

$$\int x\ln(1+x)dx = \left(\frac{1}{2}(1+x)^2 - (1+x)\right)\ln(1+x) - \frac{1}{4}(1+x)^2 + 1 + x + C$$

This answer is fine. You can simplify it a bit more for kicks and giggles:

$$\int x\ln(1+x)dx = \frac{1}{2}(x^2-1)\ln(1+x) - \frac{1}{4}x^2 + \frac{1}{2}x + C$$

□

12. $\int \sin(\ln x)dx$

Solution: Use the substitution $w = \ln x$. Then $dw = \frac{1}{x}dx \Rightarrow xdw = dx \Rightarrow e^w dw = dx$ since $x = e^w$ from our substitution. Then we have:

$$\int \sin(\ln x)dx = \int e^w \sin w dw$$

This is the same as Problem #1, so

$$\int e^w \sin w dw = \frac{1}{2}(e^w \sin w - e^w \cos w) + C$$

Plug back in w:

$$\int \sin(\ln x)dx = \frac{1}{2}\left(x\sin(\ln x) - x\cos(\ln x)\right) + C$$

□

13. $\int x^3\sqrt{1+x^2}dx$

You can do this problem a couple different ways. I will show you two solutions.

Solution I: You can actually do this problem without using integration by parts. Use the substitution $w = 1 + x^2$. Then $dw = 2xdx$ and $x^2 = w - 1$:

$$\int x^3\sqrt{1+x^2}dx = \int x\cdot x^2\sqrt{1+x^2}dx = \frac{1}{2}\int (w-1)\sqrt{w}dw = \frac{1}{2}\int (w^{3/2} - w^{1/2})dw$$

$$= \frac{1}{5}w^{5/2} - \frac{1}{3}w^{3/2} + C = \frac{1}{5}(1+x^2)^{5/2} - \frac{1}{3}(1+x^2)^{3/2} + C$$

Solution II: You can use integration by parts as well, but it is much more complicated. Rewrite the integral:

$$\int x^3\sqrt{1+x^2}dx = \int \frac{1}{2}x^2\cdot 2x\sqrt{1+x^2}dx$$

Let $u = \frac{1}{2}x^2, dv = 2x\sqrt{1+x^2}dx$. Then $du = xdx$, $v = \frac{2}{3}(1+x^2)^{3/2}$ (using a substitution on dv):

$$\int \frac{1}{2}x^2 \cdot 2x\sqrt{1+x^2}dx = \frac{1}{3}x^2(1+x^2)^{3/2} - \frac{2}{3}\int x(1+x^2)^{3/2}dx$$

You can use a substitution on the last integral:

$$\int \frac{1}{2}x^2 \cdot 2x\sqrt{1+x^2}dx = \frac{1}{3}x^2(1+x^2)^{3/2} - \frac{2}{15}(1+x^2)^{5/2} + C$$

□

14. Find the area between the given curves: $y = x^2 \ln x$, $y = 4\ln x$

Solution: We need to find when the two curves intersect, so set them equal to each other:

$$x^2 \ln x = 4\ln x \Rightarrow (x^2-4)\ln x = 0 \Rightarrow (x-2)(x+2)\ln x = 0$$

The solutions to this equation are $x = -2, 2, 1$. But, $x = -2$ isn't in our domain (since $\ln x$ has the domain $(0, \infty)$), so we are going to toss that solution out. This means we are going to integrate from $x = 1$ to $x = 2$. You can just guess which function is on the top or bottom:

$$A = \int_1^2 (\text{top function} - \text{bottom function})\, dx = \int_1^2 (4\ln x - x^2\ln x)dx = \int_1^2 (4-x^2)\ln x dx$$

Using integration by parts, let $u = \ln x, dv = (4-x^2)dx$. Then $du = \frac{1}{x}dx, v = 4x - \frac{1}{3}x^3$:

$$\int_1^2 (4-x^2)\ln x dx = \left(4x - \frac{1}{3}x^3\right)\ln x\Big|_1^2 - \int_1^2 \left(4 - \frac{1}{3}x^2\right)dx$$

$$= \left[\left(4x - \frac{1}{3}x^3\right)\ln x - 4x + \frac{1}{9}x^3\right]\Big|_1^2 = \frac{16}{3}\ln 2 - \frac{29}{9}$$

□

15. Use the method of cylindrical shells to the find the volume generated by rotating the region bounded by the given curves about the specified axis: $y = e^{-x}$, $y = 0$, $x = -1$, $x = 0$ about $x = 1$.

Solution: Draw a picture of what is happening. Recall that the volume for a cylinder is $V = 2\pi RH$. In this scenario, $R = 1 - x$ and $H = e^{-x}$ (since H is the top function minus the bottom function). x is going from -1 to 0:

$$V = \int_{-1}^0 2\pi(1-x)e^{-x}dx$$

Using tabular is pretty quick with $u = 1-x, dv = e^{-x}dx$:

$$\int_{-1}^0 2\pi(1-x)e^{-x}dx = 2\pi xe^{-x}\Big|_{-1}^0 = 2\pi e$$

□