Math 119 Recitation Week 05
Fall 2020

November 12, 2020

1. Solve the following problems:

   (a) Show that $x^3 + x^2 + 3x + 7 = 0$ has exactly one real root.

   Solution: Define $f(x) = x^3+x^2+3x+7$. Since x^3+x^2+3x+7 is a polynomial, $f(x)$ is **continuous** everywhere. If we put $x = -2$, then we get $-8+4-6+7 = -3 < 0$, and if we put $x = -1$, then we get $-1 + 1 - 3 + 7 = 4 > 0$. Therefore, by intermediate value theorem, there exists **at least** one root $c_1 \in (-2, -1)$ such that $f(c_1) = c_1^3 + c_1^2 + 3c_1 + 7 = 0$.

   Suppose that there is another root, say c_2, and without loss of generality, assume $c_2 > c_1$. Since $f(x)$ is **continuous** on the interval $[c_1, c_2]$ and $f(x)$ is **differentiable** on (c_1, c_2), we can apply **the mean value theorem**. Therefore, by mean value theorem, there exists a $c_3 \in (c_1, c_2)$ such that

   $$f'(c_3) = \frac{f(c_2) - f(c_1)}{c_2 - c_1}.$$

   Since $f(c_1) = 0, f(c_2) = 0$ and $c_2 - c_1 \neq 0$, we obtain $f'(c_3) = 0$. That is, c_3 is a root of $f'(x)$. However, the derivative $f'(x) = 3x^2 + 2x + 3$ has discriminant $\Delta = 2^2 - 4 \cdot 3 \cdot 3 < 0$. In other words, $f'(x)$ does not have any real root. This gives a **contradiction**. Thus, c_2 is not a root of $f(x)$.

   Hence, the equation $x^3 + x^2 + 3x + 7 = 0$ has exactly one real root, namely c_1.

   (b) (**Exercise**) Let $f(x) = x^4 + x^2 + x - 10$.

   i. Show that f has at least 2 real roots.

   ii. Show that f does **not** have 3 real roots or more.

2. Prove the following inequalities.

   (a) $e^x \geq x + 1$ for all x.

Solution: Define $f(x) = e^x - x - 1$. We want to prove that $f(x) \geq 0$ for all x. First, we take the derivative of $f(x)$, $f'(x) = e^x - 1$.

Case 1: $x \geq 0$

In this case, $f'(x) = e^x - 1 > 0$ for all $x > 0$, and $f'(0) = 0$. So, $f(x)$ is increasing for all $x \geq 0$. Since $f(0) = 1 - 0 + 1 = 0$ and f is increasing, we obtain $f(x) \geq 0$ for all $x \geq 0$.

Case 2: $x < 0$

In this case, since $e^x < 1$ for all $x < 0$, we have $f'(x) = e^x - 1 < 0$. So, $f(x)$ is decreasing for all $x < 0$. Since $f(0) = 0$ and $f(x)$ is decreasing for all $x \in (-\infty, 0)$, we get $f(x) > 0$ for all $x < 0$.

Thus, $f(x) = e^x - x - 1 \geq 0$ for all x.

(b) $e^x \geq 1 + x + \frac{x^2}{2}$ for all $x \geq 0$.

Solution: Define $f(x) = e^x - 1 - x - \frac{x^2}{2}$. We want to show that $f(x) \geq 0$ for all $x \geq 0$.

First, we evaluate $f(0) = 1 - 1 - 0 - 0 = 0$. Then, we take the derivative of $f(x)$, $f'(x) = e^x - 1 - x$. By the previous part, we know that $e^x \geq 1 + x$. Therefore, $f'(x) = e^x - 1 - x > 0$ for all $x > 0$ and $f'(0) = 0$. Hence, $f(x)$ is increasing for all $x \geq 0$.

Since $f(x)$ is increasing for all $x \geq 0$ and $f(0) = 0$, we conclude that $f(x) = e^x - 1 - x - \frac{x^2}{2} \geq 0$ for all $x \geq 0$.

(c) (**Exercise**) $\tan x > x$ for $0 < x < \pi/2$.

3. Solve the following problems:

(a) Let f be a function defined on $[0, 9]$ and f', f'' exist on $(0, 9)$. If $f(1) = -2$, $f(3) = 5$, $f(4) = 6$ and $f(8) = 20$, then show that there is a number $c \in (0, 9)$ such that $f''(c) = 0$.

Solution: Since f' and f'' exist on $[0, 9]$, f and f' are differentiable on $[0, 9]$.

Also, **if** f is differentiable, **then** f is continuous.

Therefore, we can use **Mean Value Theorem(MVT)** for both f and f'.

Since f is continuous on $[1, 3]$ and differentiable on $(1, 3)$, by MVT, there exists a $c_1 \in (1, 3)$ such that

$$f'(c_1) = \frac{f(3) - f(1)}{3 - 1} = \frac{(5) - (-2)}{3 - 1} = \frac{7}{2}.$$

Similarly, since f is continuous on $[4, 8]$ and differentiable on $(4, 8)$, by MVT, there exists a $c_2 \in (4, 8)$ such that

$$f'(c_2) = \frac{f(8) - f(4)}{8 - 4} = \frac{(20) - (6)}{8 - 4} = \frac{7}{2}.$$

Since f' is continuous on $[c_1, c_2] \subseteq [0, 9]$ and differentiable on $(c_1, c_2) \subseteq (0, 9)$, and $c_2 - c_1 \neq 0$, by MVT, there exists a number $c \in (c_1, c_2) \subseteq (0, 9)$ such that

$$f''(c) = \frac{f'(c_2) - f'(c_1)}{c_2 - c_1} = \frac{\frac{7}{2} - \frac{7}{2}}{c_2 - c_1} = 0.$$

(b) If $f(1) = 10$ and $f'(x) \geq 2$ for all x, then show that $f(4) \geq 16$.

Solution: Since $f'(x) \geq 2$ for all x, $f(x)$ is continuous and differentiable for all x. Hence, we can apply the Mean Value Theorem on the interval $[1, 4]$.

Since f is continuous on $[1, 4]$ and differentiable on $(1, 4)$, there exists a value $c \in (1, 4)$ such that

$$f'(c) = \frac{f(4) - f(1)}{4 - 1} = \frac{f(4) - 10}{3}.$$

Since $f'(x) \geq 2$ for all x, we get $f'(c) \geq 2$, too. Therefore,

$$f'(c) = \frac{f(4) - 10}{3} \geq 2 \implies f(4) - 10 \geq 6 \implies f(4) \geq 16.$$

4. Given $f(x) = x^5 + 7x - 2\sin(\pi x) - 2$,

(a) Show that $f^{-1}(x)$ exists.

Recall that f^{-1} exists if f is one-to-one (1-1).

Solution: The derivative of f is

$$f'(x) = 5x^4 + 7 - 2\pi\cos(\pi x).$$

Since $7 + 2\pi \geq 7 - 2\pi\cos(\pi x) \geq 7 - 2\pi > 0$, we have

$$f'(x) = 5x^4 + 7 - 2\pi\cos(\pi x) > 0.$$

Therefore, f is strictly increasing, and hence, f is 1-1. Thus, $f^{-1}(x)$ exists.

(b) Find the domain and range of $f^{-1}(x)$.

Solution: Observe that *Dom(f)*=$\mathbb{R}$ and *Range(f)*=$\mathbb{R}$. Since *Dom(f)*=*Range(f^{-1})* and *Range(f)*=*Dom(f^{-1})*, we obtain that *Dom(f^{-1})*=$\mathbb{R}$ and *Range(f^{-1})*=$\mathbb{R}$.

(c) Compute $\left.\left(\dfrac{d}{dx}f^{-1}(x)\right)\right|_{x=6}$

Solution: Observe that $y = f^{-1}(x) \iff f(y) = x$, and we want to find y'. Since $y = f^{-1}(x) \iff f(y) = x$, if we take derivative of both sides with respect to x, we get, *by chain rule*, that

$$y' \cdot f'(y) = 1.$$

Therefore, since $y = f^{-1}(x)$ (be careful, it is y, **not** y')

$$y' = \frac{1}{f'(y)} = \frac{1}{f'(f^{-1}(x))}.$$

Also, since $f^{-1}(6) = c \implies f(c) = 6$, we have

$$f(c) = c^5 + 7c - 2\sin(\pi \cdot c) - 2 = 6 \implies c = 1.$$

Thus, $f^{-1}(6) = 1$, and hence

$$y'|_{x=6} = \left.\left(\frac{d}{dx}f^{-1}(x)\right)\right|_{x=6} = \frac{1}{f'(f^{-1}(6))} = \frac{1}{f'(1)} =$$

$$= \left.\frac{1}{5x^4 + 7 - 2\pi\cos(\pi x)}\right|_{x=1} = \frac{1}{12 + 2\pi}.$$

5. Find $\dfrac{dy}{dx}$ at the point where $x = 1$ if the differentiable function $y = f(x)$ is defined by

$$2xe^y + ye^x = 3e^x, \quad f(1) = 1.$$

Solution: Since y also depends on x and we do not know explicit form of y, we will use implicit differentiation. Then, if we take derivative of all sides with respect to x, we obtain

$$2e^y + 2xy'e^y + y'e^x + ye^x = 3e^x.$$

Since $y = f(1) = 1$, we get

$$2e + 2ey' + ey' + e = 3e \implies 3ey' = 0 \implies \left.\frac{dy}{dx}\right|_{x=1} = 0.$$

6. Consider the curve given by the following implicit equation $\tan(x + y) = \sin(xy)$. Find the tangent line to this curve at the point $(\sqrt{\pi}, -\sqrt{\pi})$.

Solution: The tangent line to this curve at the point $(\sqrt{\pi}, -\sqrt{\pi})$ has the slope $\left.\dfrac{dy}{dx}\right|_{x=\sqrt{\pi}}$. Therefore, we take the derivative of all sides of $\tan(x + y) = \sin(xy)$ with respect to x. So,

$$\sec^2(x + y) \cdot (1 + y') = cos(xy) \cdot (y + xy').$$

Since $y = -\sqrt{\pi}$ when $x = \sqrt{\pi}$, we get

$$\sec^2(0)(1 + y') = cos(-\pi)(-\sqrt{\pi} + \sqrt{\pi} \cdot y') \implies 1 + y' = \sqrt{\pi} - \sqrt{\pi}y'$$

$$\implies y'(1 + \sqrt{\pi}) = \sqrt{\pi} - 1 \implies \left.\frac{dy}{dx}\right|_{x=\sqrt{\pi}} = \frac{\sqrt{\pi} - 1}{\sqrt{\pi} + 1}.$$

Thus, the tangent line to this curve at the point $(\sqrt{\pi}, -\sqrt{\pi})$ has the equation,

$$y = \frac{\sqrt{\pi} - 1}{\sqrt{\pi} + 1}(x - \sqrt{\pi}) + (-\sqrt{\pi}).$$

7. If $xy + y^3 = 1$, find the value of y'' at the point where $x = 0$.

Solution: We need to take the derivative twice. Therefore, by product and chain rule,

$$y + xy' + 3y^2y' = 0, \text{ and } y' + y' + xy'' + 6yy'y' + 3y^2y'' = 0.$$

In other words, $2y' + (x + 3y^2)y'' + 6y(y')^2 = 0$.
Observe that, if we put $x = 0$ into the equation $xy + y^3 = 1$, we get $y = 1$.
Similarly, if we put $x = 0, y = 1$ into the equation $y+xy'+3y^2y' = 0$, we obtain $y' = \frac{-1}{3}$.
Also, if we put $x = 0, y = 1$, and $y' = \frac{-1}{3}$ into the equation

$$2y' + (x + 3y^2)y'' + 6y(y')^2 = 0,$$

we get the equality $\frac{-2}{3} + 3y'' + \frac{6}{9} = 0$ which implies that $y'' = 0$, when $x = 0$.