Math 119 Recitation Week 04
Fall 2020

November 5, 2020

1. Consider the curve given by $y = \dfrac{16}{x} - x^2$. Find the points where this curve has a horizontal tangent line.

   Solution: Any horizontal line has the slope 0. So, the derivative of the function $f(x) = \frac{16}{x} - x^2$ at the point (x_0, y_0), which is $f^{'}(x_0) = y_0$, is equal to 0. That is, $f^{'}(x_0) = 0$. By direct computation, we obtain $f^{'}(x_0) = \frac{-16}{x_0^2} - 2x_0$. Hence, $f^{'}(x_0) = 0$ implies that $x_0 = -2$. Thus, the point $(-2, f(-2)) = (-2, -12)$ is the point where the curve has a horizontal tangent line.

2. Show that there are two distinct tangent lines to the curve $y = x^2$ passing through the point (a, b) provided $b < a^2$. What about if $b = a^2$ or $b > a^2$?

   Solution: Suppose that the curve has a tangent line at the point $(x_0, y_0) = (x_0, x_0^2)$. Then, the line has slope $2x_0$, which is the value of the derivative of $y = x^2$ at the point (x_0, y_0). Also, the slope of a line passing through the points (x_0, x_0^2) and (a, b) is $\frac{x_0^2 - b}{x_0 - a}$. Therefore, we get $2x_0 = \frac{x_0^2 - b}{x_0 - a}$, which implies the equation $x_0^2 - 2x_0 a + b = 0$. If we investigate the discriminant $\Delta = 4a^2 - 4b$, we obtain that $\Delta > 0$ since $a^2 > b$ is given. Hence, there exist two different x_0 values. This result implies that there exist two different tangent lines to the curve $y = x^2$ passing through the points (x_0, x_0^2) and (a, b).

   Observe that if $b = a^2$, then there will be only one x_0 value. Hence, there will be just one tangent line to the curve $y = x^2$. If $b > a^2$, then this point (a, b) will be above the curve. Therefore, there cannot be any tangent line to the curve passing through that point.

3. For what values(s) of the constant k do the curves $y = kx^2$ and $y = k(x-2)^2$ intersect at right angles?

Solution: If these two curves intersect, then the common point will be $(x_0, y_0) = (x_0, kx_0^2)$ and $(x_0, k(x_0-2)^2)$. So, we have $kx_0^2 = k(x_0-2)^2 \implies 0 = -4kx_0 + 4k$. Therefore, either $k = 0$ or $x_0 = 1$. The case that $k = 0$, does not make any sense. Hence, the common point $(x_0, y_0) = (1, k)$. Now, since these two curves intersect at right angles, the tangents to these curves at the point $(1, k)$ will be perpendicular to each other. Hence, the product of their slopes is -1. Thus, we get $2kx_0 \cdot 2k(x_0-2) = (2k)\cdot(-2k) = -1 \implies k = \pm\frac{1}{2}$.

4. Calculate the derivative of the given function *using the definition of the derivative.*

   (a) $F(x) = \dfrac{1}{\sqrt{1+x^2}}$

   (b) $f(x) = x^{1/3}$

Recall: The derivative of $f(x)$ with respect to x is the function $f^{'}(x)$ and is defined as:

$$f'(x) = \lim_{h\to 0} \frac{f(x+h) - f(x)}{h}$$

Solution:

(a) First, we find the common denominator. Then, we multiply and divide by the conjugate $(\sqrt{1+x^2} + \sqrt{1+(x+h)^2})$. After calculating the numerator, we cancel out the common h term both in numerator and in denominator. Finally, when we take limit as h goes to 0, we obtain the result.

$$F^{'}(x) = \lim_{h\to 0} \frac{\frac{1}{\sqrt{1+(x+h)^2}} - \frac{1}{\sqrt{1+x^2}}}{h} = \lim_{h\to 0} \frac{\sqrt{1+x^2} - \sqrt{1+(x+h)^2}}{h\cdot(\sqrt{[1+(x+h)^2](1+x^2)}} =$$

$$\lim_{h\to 0} \frac{(1+x^2) - (1+(x+h)^2)}{h\cdot(\sqrt{[1+(x+h)^2](1+x^2)}\cdot[\sqrt{1+x^2} + \sqrt{1+(x+h)^2}]} =$$

$$\lim_{h\to 0} \frac{-2xh - h^2}{h\cdot\sqrt{[1+(x+h)^2](1+x^2)}\cdot[\sqrt{1+x^2} + \sqrt{1+(x+h)^2}]} =$$

$$\lim_{h\to 0} \frac{-2x - h}{\sqrt{[1+(x+h)^2](1+x^2)}\cdot[\sqrt{1+x^2} + \sqrt{1+(x+h)^2}]} = \frac{-2x}{2\sqrt{(1+x^2)^3}}$$

(b) We can use the mathematical identity $a^3 - b^3 = (a-b)(a^2 + ab + b^2)$ if we take $a = (x+h)^{\frac{1}{3}}$ and $b = x^{\frac{1}{3}}$. So, we multiply and divide by $[(x+h)^{\frac{2}{3}} + (x+h)^{\frac{1}{3}} \cdot x^{\frac{1}{3}} + x^{\frac{2}{3}}]$. After some calculations, we cancel out the common "h" terms from numerator and denominator. Finally, we evaluate the limit as h goes to 0.

$$f^{'}(x) = \lim_{h \to 0} \frac{(x+h)^{\frac{1}{3}} - x^{\frac{1}{3}}}{h} = \lim_{h \to 0} \frac{x+h-x}{h \cdot [(x+h)^{\frac{2}{3}} + (x+h)^{\frac{1}{3}} \cdot x^{\frac{1}{3}} + x^{\frac{2}{3}}]} =$$

$$\lim_{h \to 0} \frac{h}{h \cdot [(x+h)^{\frac{2}{3}} + (x+h)^{\frac{1}{3}} \cdot x^{\frac{1}{3}} + x^{\frac{2}{3}}]} = \lim_{h \to 0} \frac{1}{[(x+h)^{\frac{2}{3}} + (x+h)^{\frac{1}{3}} \cdot x^{\frac{1}{3}} + x^{\frac{2}{3}}]} = \frac{1}{3x^{\frac{2}{3}}}$$

5. How should the function $f(x) = x^2 \sin(\frac{1}{x})$ be defined at $x = 0$ so that it is continuous at $x = 0$? Is it then differentiable there?

Solution: For continuity, the limit of $f(x)$ at $x = 0$ should be equal to the value $f(0)$. So, we first evaluate the limit at $x = 0$. By squeeze theorem, it is clear that

$$\lim_{x \to 0} f(x) = \lim_{x \to 0} x^2 sin(\frac{1}{x}) = 0.$$

Therefore, **f(x) should be defined as 0 at the point where $\mathbf{x = 0}$**. Then, $f(x)$ becomes continuous at $x = 0$.

Recall that the derivative of $f(x)$ at $x = a$ is $f^{'}(a) = \lim_{x \to a} \frac{f(x) - f(a)}{x - a}$. Therefore, the derivative of $f(x)$ at $x = 0$ is (by squeeze theorem)

$$f^{'}(0) = \lim_{x \to 0} \frac{f(x) - f(0)}{x - 0} = \lim_{x \to 0} \frac{x^2 sin(\frac{1}{x}) - 0}{x - 0} = \lim_{x \to 0} x sin(\frac{1}{x}) = 0.$$

Since $f^{'}(0)$ **exists**, the function $f(x)$ is differentiable at $x = 0$.

6. Let $g(x)$ be continuous at $x = a$ and consider the function $f(x) = (x-a)g(x)$. Find $f'(a)$ in terms of g.

Solution: Since $g(x)$ is continuous at $x = a$, we have $\lim_{x \to a} g(x) = g(a)$. Also, we have $f(a) = (a - a) \cdot g(a) = 0$.

Notice that we do not know whether $g'(x)$ exists. Therefore, we cannot use differentiation formulas. In other words, we have to use the definition of derivative at $x = a$. Since $f(a) = 0$ and $g(x)$ is continuous at $x = a$, we have

$$f'(a) = \lim_{x \to a} \frac{(x-a)g(x) - 0}{x - a} = \lim_{x \to a} g(x) = g(a).$$

Thus, $f'(a) = g(a)$.

7. Given that $f(1) = 2, f'(1) = 1, g(1) = 3, g'(1) = 4$, calculate the following:

(a) $\frac{d}{dx}\left(\frac{f(x)}{g(x)+x}\right)\Big|_{x=1}$

(b) $\frac{d}{dx}(x^3 f(x))\Big|_{x=1}$

(c) $\frac{d}{dx}(f^2(x)g(x))\Big|_{x=1}$

Solution:

(a) By Quotient rule,
$$\frac{d}{dx}\left(\frac{f(x)}{g(x)+x}\right)\Big|_{x=1} = \left(\frac{f'(x)[g(x)+x] - [g'(x)+1]f(x)}{[g(x)+x]^2}\right)\Big|_{x=1} =$$
$$= \frac{f'(1)[g(1)+1] - [g'(1)+1]f(1)}{[g(1)+1]^2} = \frac{1\cdot 4 - 5\cdot 2}{4^2} = \frac{-6}{16}$$

(b) By Product rule,
$$\frac{d}{dx}(x^3 f(x))\Big|_{x=1} = \left(3x^2 f(x) + x^3 f'(x)\right)\Big|_{x=1} = 3\cdot 2 + 1\cdot 1 = 7$$

(c) By product and chain rule,
$$\frac{d}{dx}(f^2(x)g(x))\Big|_{x=1} = \left(2\cdot f(x)\cdot f'(x)\cdot g(x) + f^2(x)\cdot g'(x)\right)\Big|_{x=1} = 2{\cdot}2{\cdot}1{\cdot}3{+}2^2{\cdot}4 = 28$$

8. Find the derivative of the following functions:

(a) $f(x) = \left(\frac{1+\sin 3x}{3-2x}\right)^{-1}$

(b) $g(x) = \tan\frac{\pi}{\sqrt{25-x^2}}$

(c) If $f'(x) = \sin(x^2)$ and $y = f\left(\frac{2x-1}{x+1}\right)$, find y'.

(d) **(Exercise)** Evaluate $\frac{dy}{dx}|_{x=1/6}$ if $y = \sin(3\pi\sin(2\pi\sin(\pi x)))$.

(e) **(Exercise)** Let $f(x) = \sqrt{x+2}$, find a formula for $f^{(n)}(x)$ and use it to find $f^{(100)}(0)$

Solution:

(a) By chain and quotient rule,

$$f'(x) = (-1) \cdot \Big(\frac{1+\sin 3x}{3-2x}\Big)^{-2} \cdot \Big(\frac{3cos(3x)(3-2x)-(-2)(1+sin(3x))}{(3-2x)^2}\Big)$$

(b) By chain rule,

$$g'(x) = \sec^2(\frac{\pi}{\sqrt{25-x^2}}) \cdot \pi \cdot \frac{-1}{2}(25-x^2)^{\frac{-3}{2}} \cdot (-2x)$$

(c) By chain rule and quotient rule,

$$y' = f'(\frac{2x-1}{x+1}) \cdot \Big(\frac{2(x+1)-1(2x-1)}{(x+1)^2}\Big) = \sin[\Big(\frac{2x-1}{x+1}\Big)^2] \cdot \Big(\frac{2(x+1)-1(2x-1)}{(x+1)^2}\Big)$$

9. a) Suppose f is a differentiable function and $y = x/4 - 3$ is an equation for the tangent line to the graph of $y = f(x)$ at the point $x = 8$. If $g(x) = (f(x^3))^2$, find an equation for the tangent line to the graph of $y = g(x)$ at the point $x = 2$.

b) **(Exercise)** If $g''(2) = 0$, find $f''(8)$.

Solution:

(a) Since the line $y = \frac{x}{4} - 3$ is tangent to the graph of $y = f(x)$ at $x = 8$, we have the slope $\frac{1}{4} = f'(8)$ and the common point $(8, \frac{8}{4} - 3) = (8, -1)$. So, $f(8) = -1$. Then, by chain rule,

$$g'(x)\Big|_{x=2} = \Big(2 \cdot f(x^3) \cdot f'(x^3) \cdot 3x^2\Big)\Big|_{x=1} = 2 \cdot f(8) \cdot f'(8) \cdot 12 = 24 \cdot -1 \cdot \frac{1}{4} = -6.$$

The slope of the tangent line to the graph $y = g(x)$ at the point where $x = 2$ is $g'(2) = -6$. Also, we have $g(2) = (f(8))^2 = (-1)^2 = 1$. Hence, the tangent line has the equation as

$$y = -6 \cdot (x-2) + 1 = -6x + 13.$$