Math 119- Recitation - Week 3 30/10/20

1) Find m so that $g(x) = \begin{cases} x-m & \text{if } x<3 \\ 1-mx & \text{if } x \geq 3 \end{cases}$

is continuous for all x.

Solution: For $x<3$, $g(x) = x-m$ is a polynomial. It is already cont.
For $x>3$, $g(x) = 1-mx$ is a poly. So, it is already continuous

For $x=3$

Recall: If $\lim_{x\to 3} g(x) = g(3)$, then $g(x)$ is cont. at $x=3$

So, $\lim_{x\to 3^-} g(x) = 3-m \;=\!=\; \lim_{x\to 3^+} g(x) = \lim_{x\to 3^+}(1-mx) = 1-3m$ (For limit)

For continuity:

$$\lim_{x\to 3} g(x) = g(3) = 1-3m$$

$\therefore$ $3-m = 1-3m \Rightarrow \boxed{m=-1}$

(If $m=-1$, then $g(x)$ is cont. for all x)

2) Assume that f is a real valued continuous function such that $\lim_{x\to 0}\left[f(x)\cos^2\left(\frac{\pi}{x}\right)\right] = 0$. Find $f(0)$.

Soln.:

Since f is cont. everywhere, f is cont. at $x=0$, too.

So, we have $\lim_{x\to 0} f(x) = f(0)$ (limit at $x=0$ exists and it is equal to $f(0)$)

Now, assume that $\lim_{x\to 0} f(x) = f(0) = L \cancel{\neq 0}$ (Assumption: $f(0) \neq 0$)

Then, since both limits $\lim_{x\to 0}\left[f(x)\cos^2\left(\frac{\pi}{x}\right)\right]$ and $\lim_{x\to 0} f(x)$ exist,

we can use limit rules.

Hence, $$\lim_{x\to 0}\left[\frac{f(x)\cos^2\left(\frac{\pi}{x}\right)}{f(x)}\right] = \frac{\lim_{x\to 0} f(x)\cos^2\left(\frac{\pi}{x}\right)}{\lim_{x\to 0} f(x)} = \frac{0}{L} = 0$$ (Since $L \neq 0$)

However, we know that $$\lim_{x\to 0}\left(\frac{f(x)\cos^2\left(\frac{\pi}{x}\right)}{f(x)}\right) = \lim_{x\to 0}\left[\cos^2\left(\frac{\pi}{x}\right)\right] = \text{d.n.e.}$$

CONTRADICTION

So, this is a contradiction. Hence, our assumption is wrong.

Therefore, we conclude that $f(0) = 0$

3) Show that there is some a with $0 < a < 2$ s.t. $a^2 + \cos(\pi a) = 4$

Recall: Intermediate Value Theorem (IVT)

If $y = f(x)$ is cont. on interval $[a,b]$ & $f(a) = c$, $f(b) = d$

then there is at least one value $m \in (a,b)$ s.t. $f(m) = n$ for some $n \in (c,d)$

Solution:

Define $f(x) = x^2 + \cos(\pi x)$. Since f is sum of a polynomial and a trigonometric func., f is cont. everywhere.

Since f is cont. on $[0,2]$, we can use IVT on $[0,2]$

For $x=0$, $f(0)=0+\cos(0)=1<4$
For $x=2$, $f(2)=4+\cos(2\pi)=5>4$

$\left(\begin{array}{l} 4\in(1,5) \\ \text{That is } 1<4<5 \end{array}\right)$

$\therefore$ By IVT, there is some $a\in(0,2)$ s.t. $f(a)=a^2+\cos(\pi a)=4$

4) Show that the equation $x^3-30x+2=0$ has at least three solutions.

Solution: Define $f(x)=x^3-30x+2$. Since f is a polynomial, f is continuous on $\mathbb{R}$. So, we can use IVT.

We want to show that there exist at least three values c_1, c_2 and c_3 such that $f(c_1)=f(c_2)=f(c_3)=0$

(i) For $x=0$, $f(0)=2>0$
For $x=1$, $f(1)=-27<0$
by IVT, $\exists$ (there exists) $c_1\in(0,1)$ such that $f(c_1)=0$

(ii) For $x=5$, $f(5)=125-150+2<0$
For $x=6$, $f(6)=216-180+2>0$
by IVT, $\exists$ a $c_2\in(5,6)$ s.t. $f(c_2)=0$

(iii) For $x=-6$, $f(-6)=-216+180+2<0$
For $x=-5$, $f(-5)=-125+150+2>0$
by IVT, $\exists$ $c_3\in(-6,-5)$ s.t. $f(c_3)=0$

5) Use the formal definition of the limit to verify the following:

(a) $\lim_{x\to c}(ax+b)=ac+b$ (b) $\lim_{x\to 2}\left(\frac{x-2}{1+x^2}\right)=0$ (c) $\lim_{x\to 3}\sqrt{2x+3}=3$

Recall: Formal Defn. of Limit

Let $f(x)$ be a func. defined on an interval that contains $x=a$, except possibly at $x=a$. Then, we say that

$$\lim_{x\to a} f(x) = L$$

if for every $\varepsilon>0$, we can find a $\delta>0$ such that

$$|f(x)-L|<\varepsilon \quad \text{whenever} \quad 0<|x-a|<\delta$$

Solutions:

(a) $\lim\limits_{x\to c}(ax+b) = ac+b$

Given $\varepsilon>0$. Choose a $\delta = \frac{\varepsilon}{|a|} > 0$.

Assume $0<|x-c|<\delta$. Then,

$$|f(x)-L| = |(ax+b)-(ac+b)| = |ax-ac| = |a(x-c)| = |a|\cdot|x-c| < |a|\cdot\delta = \varepsilon$$

Choose $\delta = \frac{\varepsilon}{|a|}$

(b) $\lim\limits_{x\to 2}\left(\frac{x-2}{1+x^2}\right) = 0$

Given $\varepsilon>0$. Choose a $\delta = \dots > 0$ such that if $0<|x-2|<\delta$ then $\left|\frac{x-2}{1+x^2}-0\right|<\varepsilon$.

1st way: Suppose $0<|x-2|<\delta$. Then,

$$\left|\frac{x-2}{1+x^2}-0\right| = \frac{1}{|1+x^2|}\cdot|x-2| \leq 1\cdot|x-2| = |x-2| < \delta = \varepsilon$$

$\left(\text{Since } 1+x^2\geq 1,\ \frac{1}{1+x^2}\leq 1\right)$ Choose $\delta=\varepsilon$.

2^{nd} way: Suppose $0<|x-a|<\delta$.

In general, when we find a δ that works, then all smaller values of δ also work.

Therefore, assume $\delta \leq 1$.

So, $|x-2|<\delta\leq 1 \Rightarrow |x-2|<1 \Rightarrow -1<x-2<1 \Rightarrow 2<1+x^2<10$

$$2<1+x^2<10 \Rightarrow \frac{1}{10}<\frac{1}{1+x^2}<\frac{1}{2}$$

So, $\left|\frac{x-2}{1+x^2}\right| = \frac{1}{|1+x^2|}\cdot|x-2| < \frac{1}{2}|x-2| < \frac{1}{2}\delta = \varepsilon \Rightarrow \delta = 2\varepsilon$

Therefore, choose $\delta = \min\{1, 2\varepsilon\}$

(c) $\lim\limits_{x\to 3}\sqrt{2x+3}=3$

Given $\varepsilon>0$. Choose a $\delta = \ldots > 0$ s.t. if $0<|x-3|<\delta$ then $|\sqrt{2x+3}-3|<\varepsilon$.

1^{st} way:

Suppose $0<|x-3|<\delta$. Then,

$$|\sqrt{2x+3}-3| = \left|\frac{(\sqrt{2x+3}-3)(\sqrt{2x+3}+3)}{\sqrt{2x+3}+3}\right| = \frac{|2x-6|}{|\sqrt{2x+3}+3|} < |2x-6| = 2|x-3| < 2\delta = \varepsilon$$

Since $\sqrt{2x+3}+3>1$, we get $\frac{1}{\sqrt{2x+3}+3}<1$

So, choose $\delta = \frac{\varepsilon}{2}$

2nd way: Suppose $0<|x-3|<\delta$.

Then,
$$\left|\sqrt{2x+3}-3\right| = \left|\frac{2x-6}{\sqrt{2x+3}+3}\right| = |2x-6|\cdot\left|\frac{1}{\sqrt{2x+3}+3}\right|$$

Assume $\delta\leq 1$. $\Rightarrow |x-3|<\delta\leq 1 \Rightarrow -1<x-3<1 \Rightarrow 2<x<4 \Rightarrow$

$$\Rightarrow \sqrt{7}<\sqrt{2x+3}<\sqrt{11} \Rightarrow \boxed{\sqrt{7}+3<\sqrt{2x+3}+3<\sqrt{11}+3}$$

So, $\boxed{\dfrac{1}{\sqrt{11}+3}<\dfrac{1}{\sqrt{2x+3}+3}<\dfrac{1}{\sqrt{7}+3}}$

Therefore,
$$|2x-6|\cdot\frac{1}{\left|\sqrt{2x+3}+3\right|}<2|x-3|\cdot\frac{1}{\left|\sqrt{7}+3\right|}<\frac{2\delta}{\sqrt{7}+3}=\varepsilon$$

$$\delta=\frac{\varepsilon(\sqrt{7}+3)}{2}$$

$\therefore$ Choose $\delta=\min\left\{1, \dfrac{\varepsilon(\sqrt{7}+3)}{2}\right\}$