Metu Math 119 - Calculus with Analytic Geometry
Recitation Week 1

1) Solve the following inequalities:

(a) $\dfrac{1}{2x+1} \geq 1-x$ (b) $|x+3|-2>3x$ (c) $\dfrac{x^3-x^2+4}{x+3} \leq 1$

Soln.:

(a) $\dfrac{1}{2x+1} - 1 + x \geq 0 \Rightarrow \dfrac{1-2x-1+2x^2+x}{2x+1} \geq 0 \Rightarrow \dfrac{2x^2-x}{2x+1} \geq 0$

Important points are $-\frac{1}{2}$, 0, and $\frac{1}{2}$

		$-\frac{1}{2}$		0		$\frac{1}{2}$	
$x(2x-1)$	$+$		$+$	0	$-$	0	$+$
$\frac{x(2x-1)}{2x+1}$	$-$		$+$	0	$-$	0	$+$

Therefore, the solution set is $\{x \in \mathbb{R} : -\frac{1}{2} < x \leq 0,\ x \geq \frac{1}{2}\}$

OR $(-\frac{1}{2}, 0] \cup [\frac{1}{2}, \infty)$

b) $|x+3| = \begin{cases} x+3 & ; \text{if } x \geq -3 \\ -x-3 & ; \text{if } x < -3 \end{cases}$

Hence, $|x+3|-2>3x \Longrightarrow$
- $x+3-2>3x$ if $x \geq -3$
- $-x-3-2>3x$ if $x<-3$

So,
- $\frac{1}{2} > x$ if $x \geq -3 \Rightarrow -3 \leq x < \frac{1}{2}$
- $-\frac{5}{4} > x$ if $x<-3 \Rightarrow$ True for all $x<-3$

Thus, the Soln. set $= (-\infty, -3) \cup [-3, \frac{1}{2}) = (-\infty, \frac{1}{2})$

(c) $\dfrac{x^3-x^2+4}{x+3} \leq 1 \Rightarrow \dfrac{x^3-x^2+4}{x+3} - 1 \leq 0 \Rightarrow \dfrac{x^3-x^2-x+1}{x+3} \leq 0$

$\Rightarrow \dfrac{x^2(x-1)-(x-1)}{x+3} \leq 0 \Rightarrow \dfrac{(x-1)(x^2-1)}{x+3} \leq 0 \Rightarrow \dfrac{(x-1)(x-1)(x+1)}{x+3} \leq 0$

$x=-1, x=1$ make it zero, and $x=-3$ makes it undefined.

Then,

		-3		-1		1	
$(x-1)^2(x+1)$	$-$		$-$	0	$+$	0	$+$
$\dfrac{(x-1)^2(x+1)}{x+3}$	$+$		$-$	0	$+$	0	$+$

when $x=0$, the result becomes 0.

$\therefore$ (Therefore) The soln. set is $\{x \in \mathbb{R} : -3 < x \leq -1, x = 1\} =$

$= (-3,-1] \cup \{1\}$

2) Write an equation for the line through the points (-1,5) and (0,3).

Recall: A line passing through (x_1, y_1), (x_2, y_2) has slope $m = \dfrac{y_2 - y_1}{x_2 - x_1}$ eqn. $y = m(x - x_1) + y_1$

So, we have slope $m = \dfrac{5-3}{-1-0} = -2$.

Then, the line $\ell : y = -2(x+1) + 5$

exc!

(3) Find the eqn. for the (a) vertical line, and (b) the horizontal line through the point $(-1, 0)$.

4) Find the eqn. for the line through the point P(-1,3) that is perpendicular to the line y+x+2=0. Find the x and y-intercepts of this line.

Recall: $\ell_1 \perp \ell_2 \underset{\text{(if and only if)}}{\Longleftrightarrow} m_1 \cdot m_2 = -1$

Also, a line $y = \underline{m} \cdot x + n$ has slope "m"

So, $y+x+2=0 \Rightarrow y=-x-2$ has slope "-1".

Then, $(-1)\cdot m = -1 \Rightarrow \boxed{m=1}$ & $\boxed{P(-1,3)}$

$\therefore$ The line has eqn. $\ell : y = 1(x+1)+3 = x+4$

To find x-intercept, put $y=0$. So, $0=x+4 \Rightarrow$ The pt. is $(-4,0)$

To find y-intercept, put $x=0$. So, $y=0+4 \Rightarrow$ The pt. is $(0,4)$

5) Describe and sketch the regions defined by the followings:

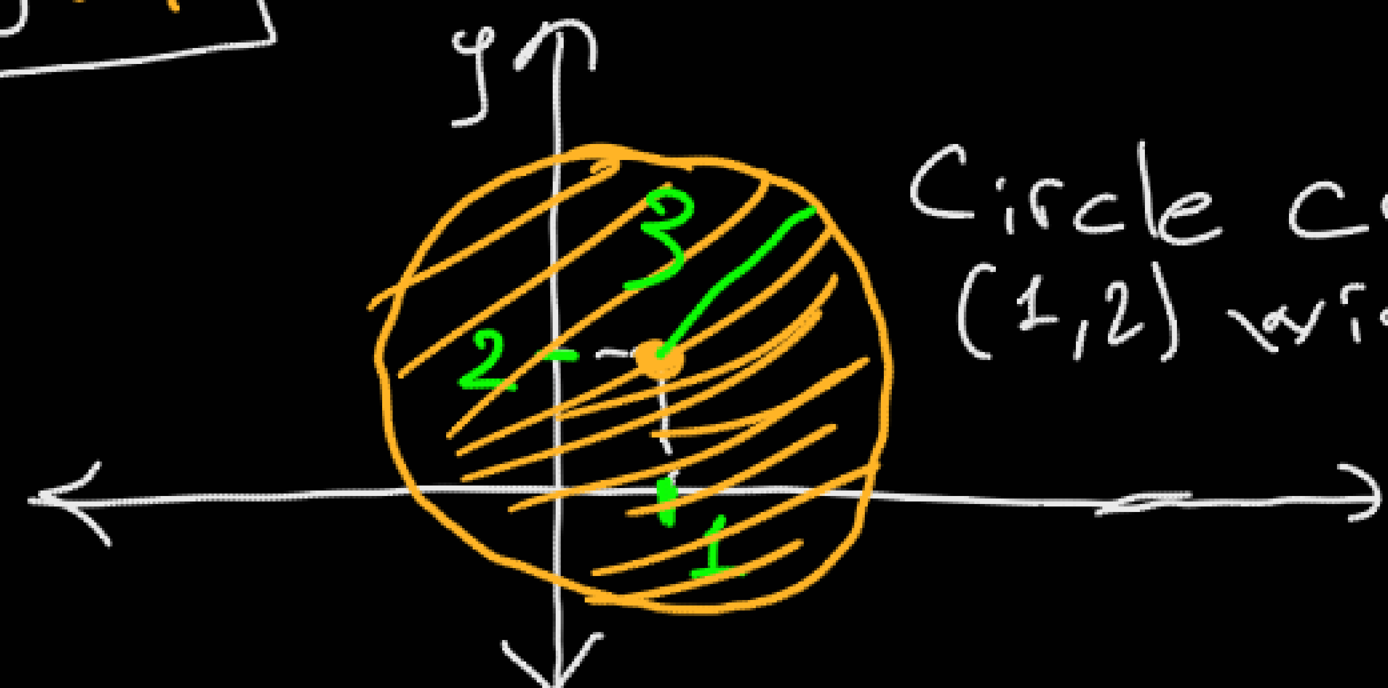
(a) $x^2+y^2-2x-4y \le 4$, (b) $x^2+y^2 \le 4$, $x^2+y^2 > 2y$,

(exc) (c) $x^2+y^2 > 2y$, $y > 1+x$, (d) $y > (x-1)^2+2$, $y < 2x$,

(e) $4x^2+(y-2)^2 \le 4$

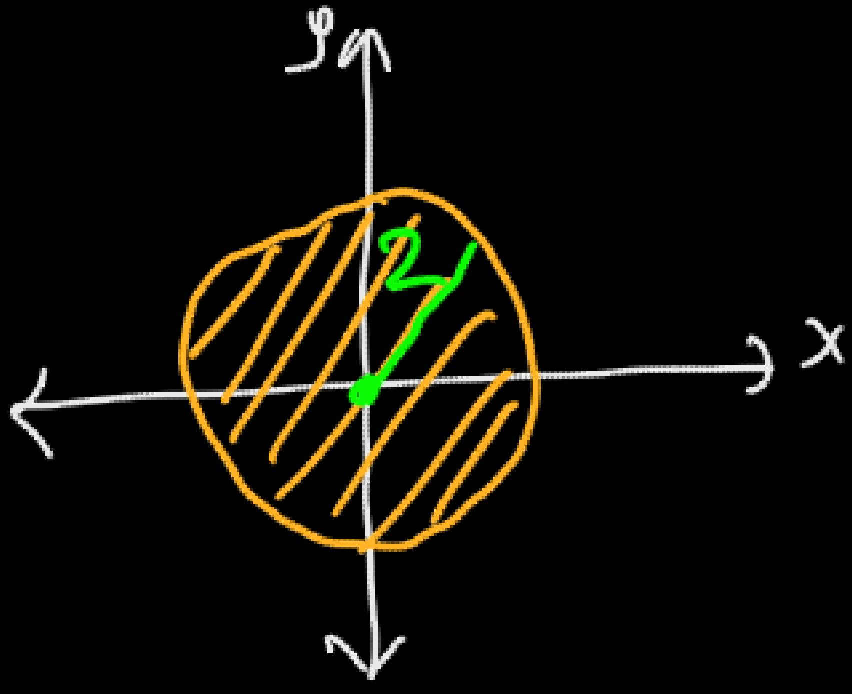
Soln.:

(a) $\underbrace{x^2-2x+1}+\underbrace{y^2-4y+4}-1-4 \le 4 \Rightarrow (x-1)^2+(y-2)^2 \le 9 = 3^2$



Circle centered at (1,2) with radius "3"

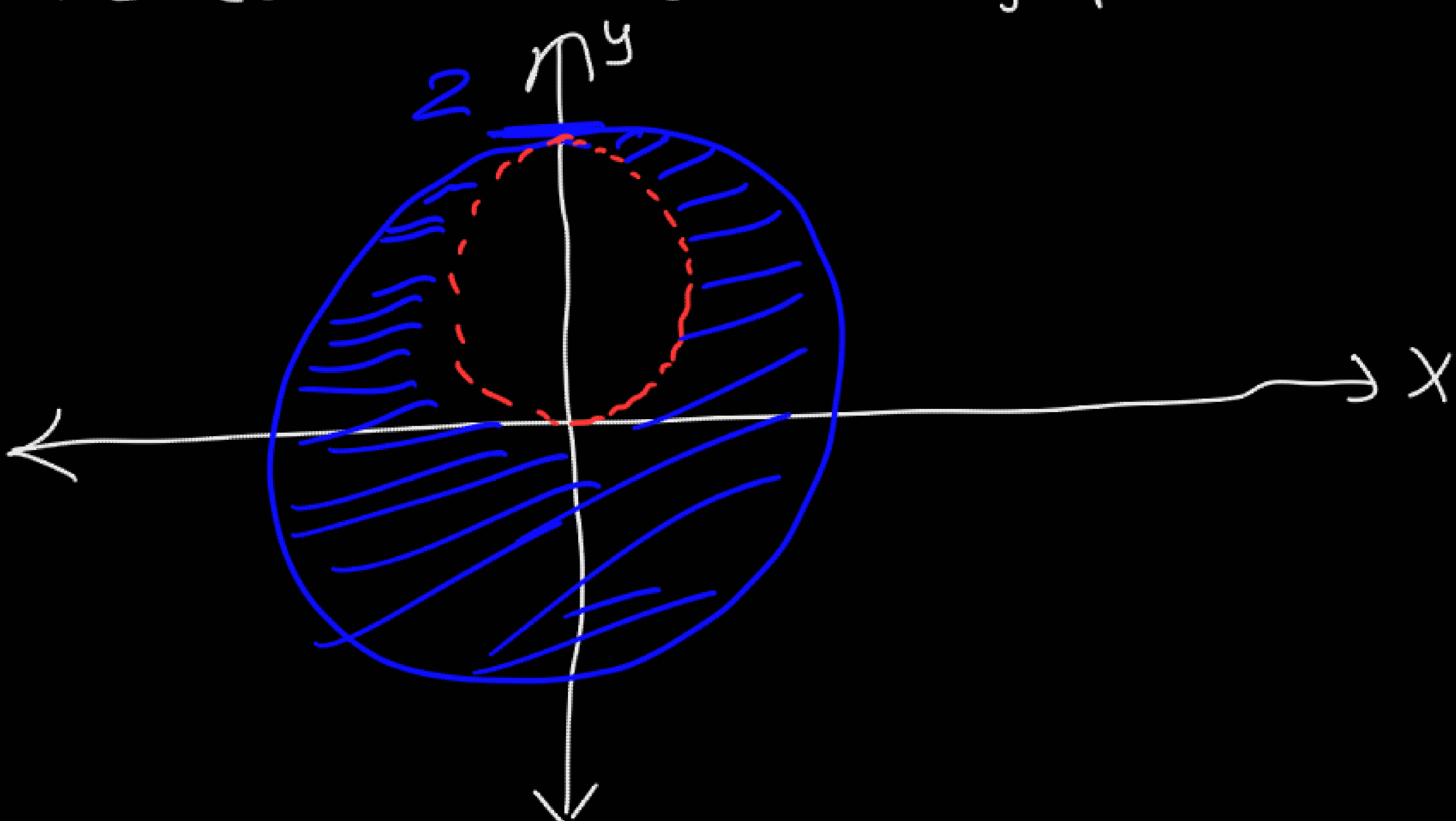
(b) $x^2+y^2 \leq 4$, $x^2+y^2>2y$



$x^2+y^2-2y+1-1>0 \Leftrightarrow (x-0)^2+(y-1)^2>1$

Center $(0,1)$, $r=1$

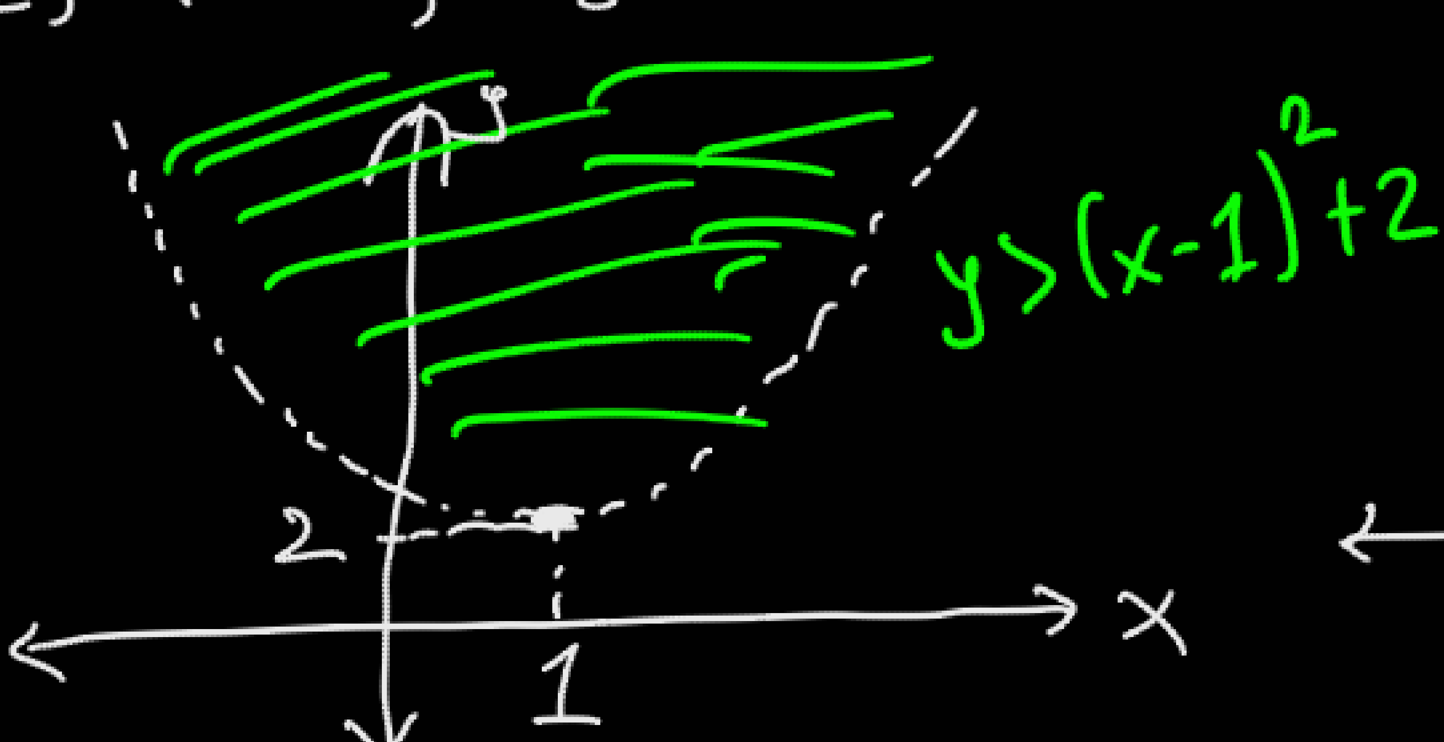
If we combine these two graphs, we obtain

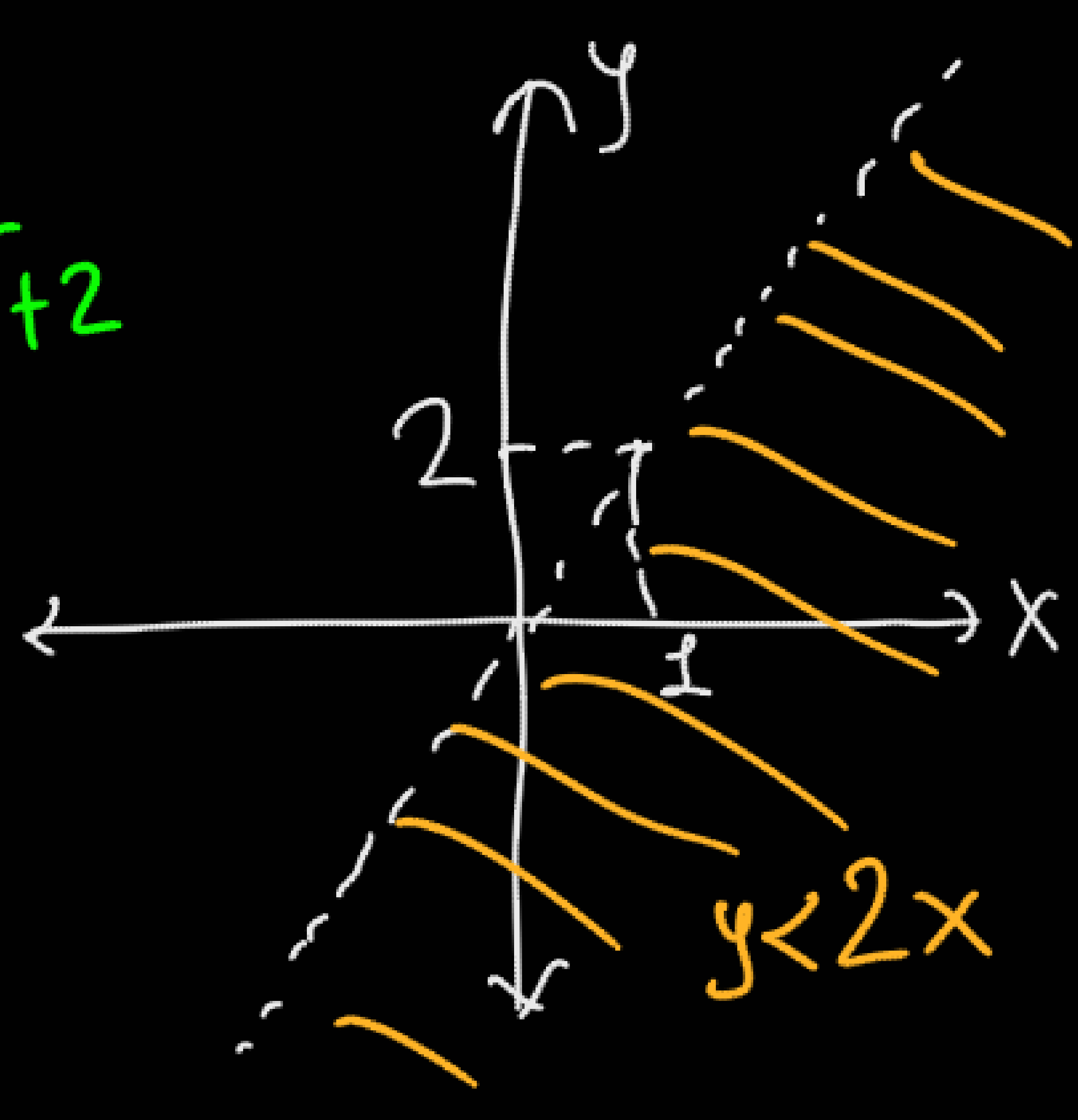


(d) $y>(x-1)^2+2$, $y<2x$

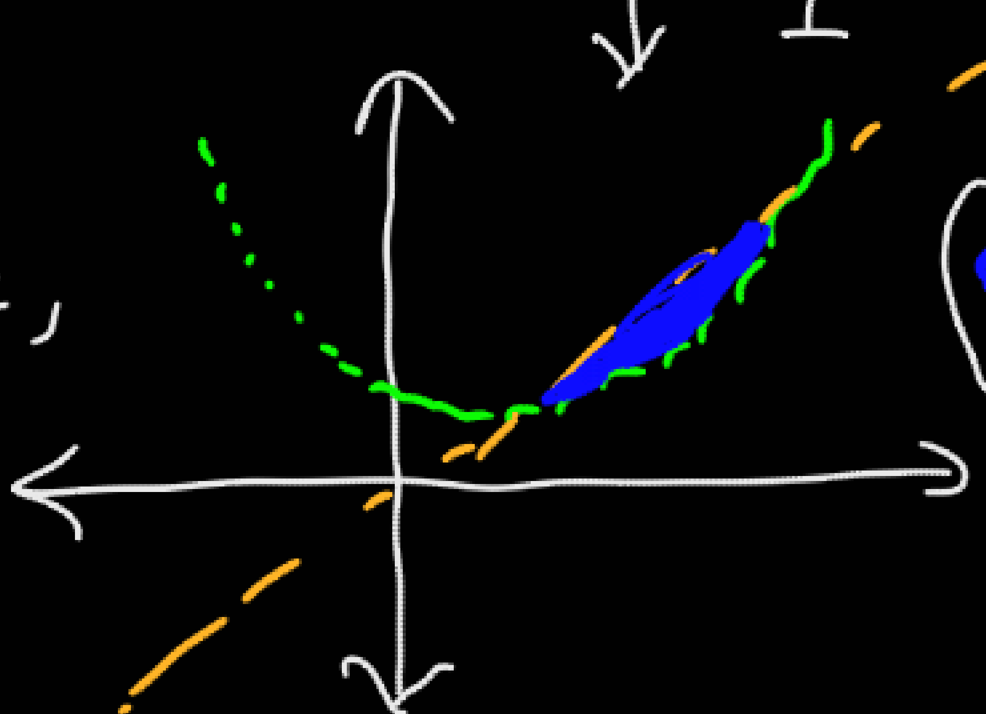
$x=1 \Rightarrow y=2$

$x=0 \Rightarrow y=3$





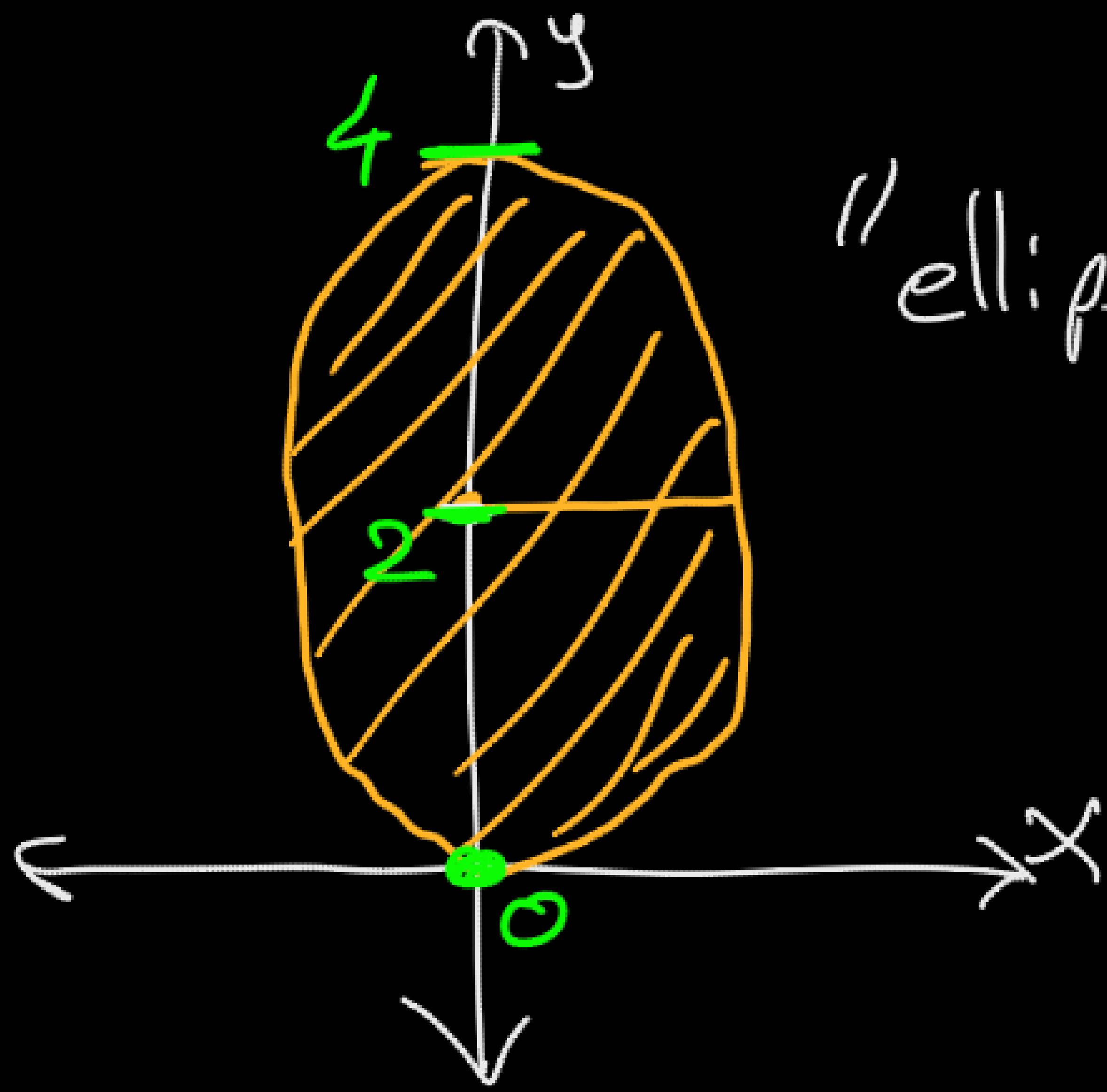
Therefore,



(is the region)

(e) $4x^2+(y-2)^2 \leq 4$

$$\frac{x^2}{1}+\frac{(y-2)^2}{2^2} \leq 1 \Rightarrow$$



6) Find the points of intersection of the pairs of curves

(a) $y=x^2+3$, $y=3x+1$, (b) $2x^2+2y^2=5$, $xy=1$

Soln.:

(a) Solve together to obtain:

$y=x^2+3=3x+1 \Rightarrow x^2-3x+2=0 \Rightarrow (x-2)(x-1)=0$

$x_1=2, \ x_2=1$

$x_1=2 \Rightarrow y_1=6+1=7$

$x_2=1 \Rightarrow y_2=1+3=4$

$\Rightarrow$ The points are $(2,7)$ & $(1,4)$

(b) $2x^2+2y^2=5$ & $xy=1 \Leftrightarrow y=\frac{1}{x}$

So, $2x^2+2\left(\frac{1}{x}\right)^2=5 \Rightarrow 2x^2+\frac{2}{x^2}=5$ multiply both sides by "x^2"

So, $2x^4-5x^2+2=0 \Rightarrow (2x^2-1)(x^2-2)=0$

$2x^2 \quad -1$

$x^2 \quad -2$

So, $x_{1,2}=\pm\sqrt{\frac{1}{2}}$ & $x_{3,4}=\pm\sqrt{2}$

Since $y=\frac{1}{x}$, $y_{1,2}=\pm\sqrt{2}$ & $y_{3,4}=\pm\frac{1}{\sqrt{2}}$

$\therefore$ The pts. are $\left(\frac{1}{\sqrt{2}},\sqrt{2}\right),\left(\frac{-1}{\sqrt{2}},-\sqrt{2}\right),\left(\sqrt{2},\frac{1}{\sqrt{2}}\right)$ and $\left(-\sqrt{2},\frac{-1}{\sqrt{2}}\right)$.

(exc) 7) Write an eqn. of the graph obtained by shifting the graph of $y = \sqrt{x}$

(a) down 1, right 1 (b) down 2, left 4

(c) up 2, left 1 (d) up 1, right 1

8) Find the domain and range of each function and sketch their graphs:

(a) $f(x)=\sqrt{x^2-1}$, (b) $f(x)=1+\sin\left(x+\frac{\pi}{4}\right)$, (c) $f(x)=\frac{1}{|2-x|}$

Soln.:

Sketching the graphs are exc. (We did some in recitation)

(a) $f(x)=\sqrt{x^2-1}$ is defined for all $x^2-1\geq 0 \iff (x-1)(x+1)\geq 0$

So,

		-1		1	
x^2-1	$+$	0	$-$	0	$+$

Hence, $\text{Dom} f=(-\infty,-1]\cup[1,\infty)$

$\text{Range} f=[0,\infty)$

(b) Since sine function is defined for all $x\in\mathbb{R}$, we see that $f(x)=1+\sin\left(x+\frac{\pi}{4}\right)$ is also defined for all $x\in\mathbb{R}$.

Hence, $\boxed{\text{Dom} f=\mathbb{R}=(-\infty,\infty)}$

Since $\sin\left(1+\frac{\pi}{4}\right)\in[-1,1]$, we have $1+\sin\left(x+\frac{\pi}{4}\right)\in[0,2]$

Hence, $\boxed{\text{Range} f=[0,2]}$

(c) $f(x)=\frac{1}{|2-x|}$

Since $|2-x|$ is in denominator, $|2-x|\neq 0$

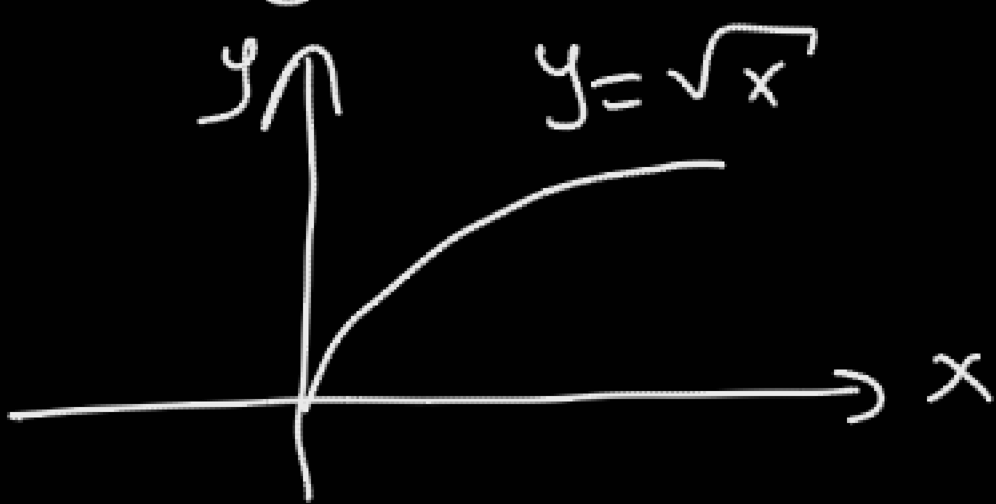
$\boxed{\text{Dom} f=\mathbb{R}\setminus\{2\}}$

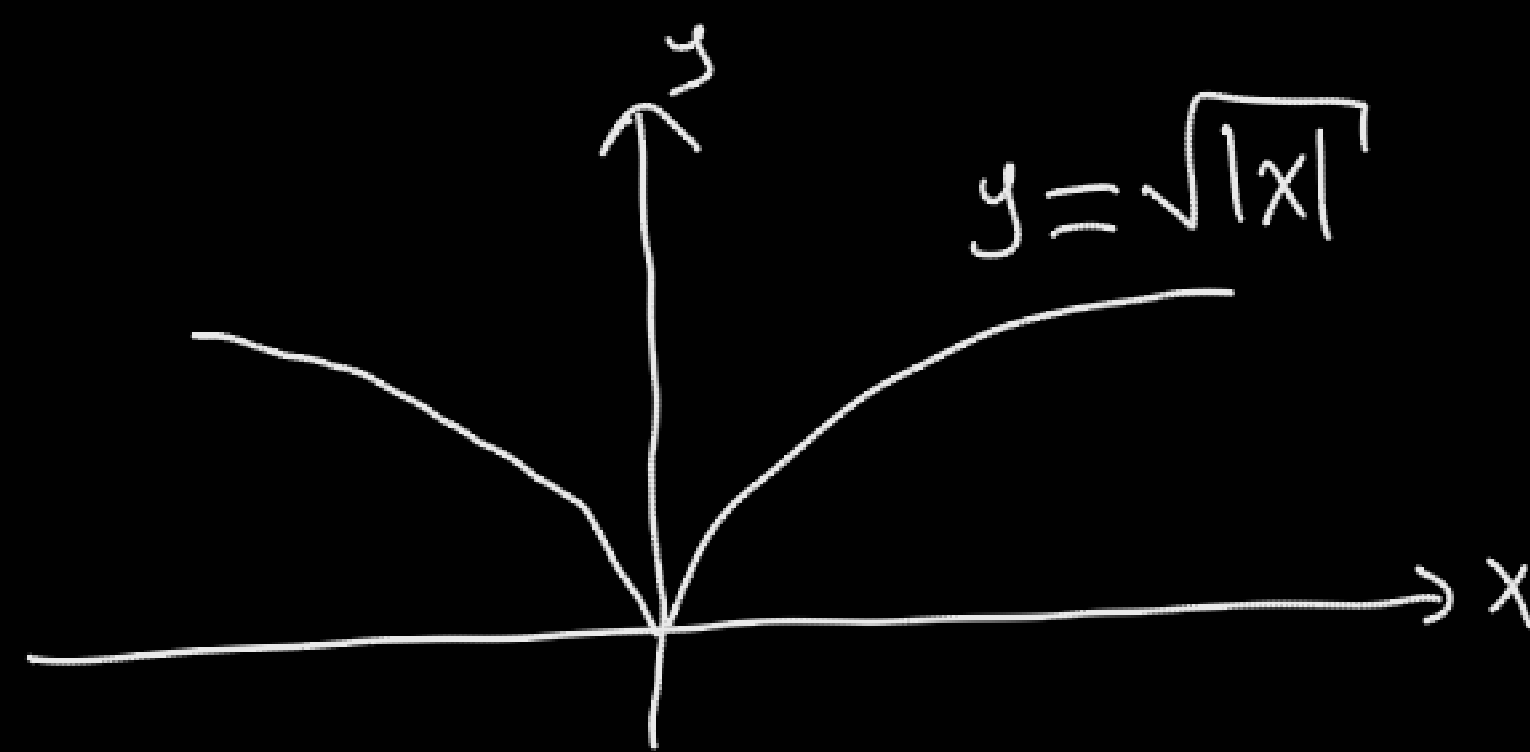
So, $|2-x|>0$. Hence, $\boxed{\text{Range} f=(0,\infty)}$

(9) (a) How is the graph of y=f(|x|) related to the graph of y=f(x) ?
(b) Sketch the graph of y= sin(|x|).
(c) Sketch the graph of $y=\sqrt{|x|}$.

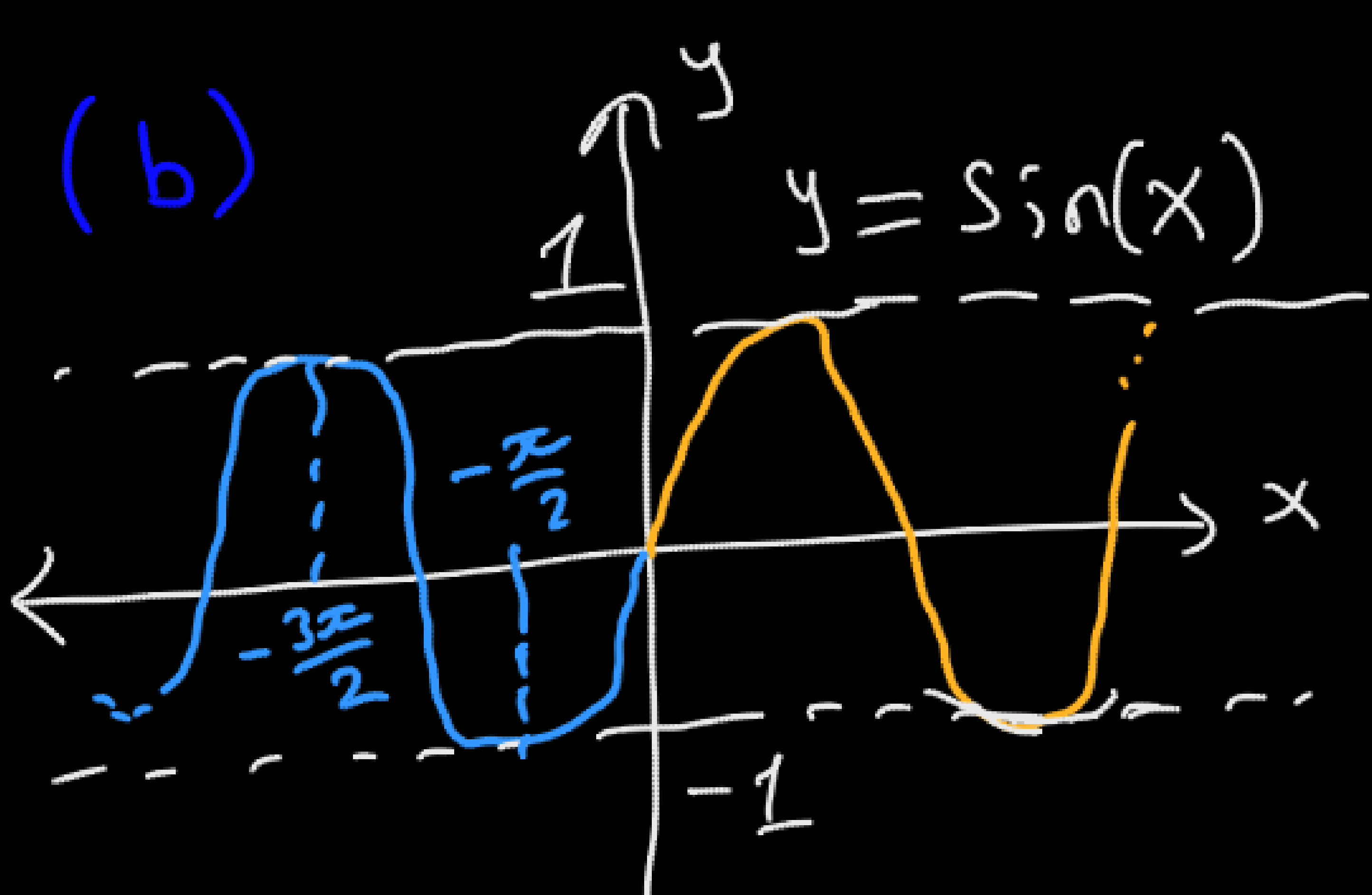
Soln.:

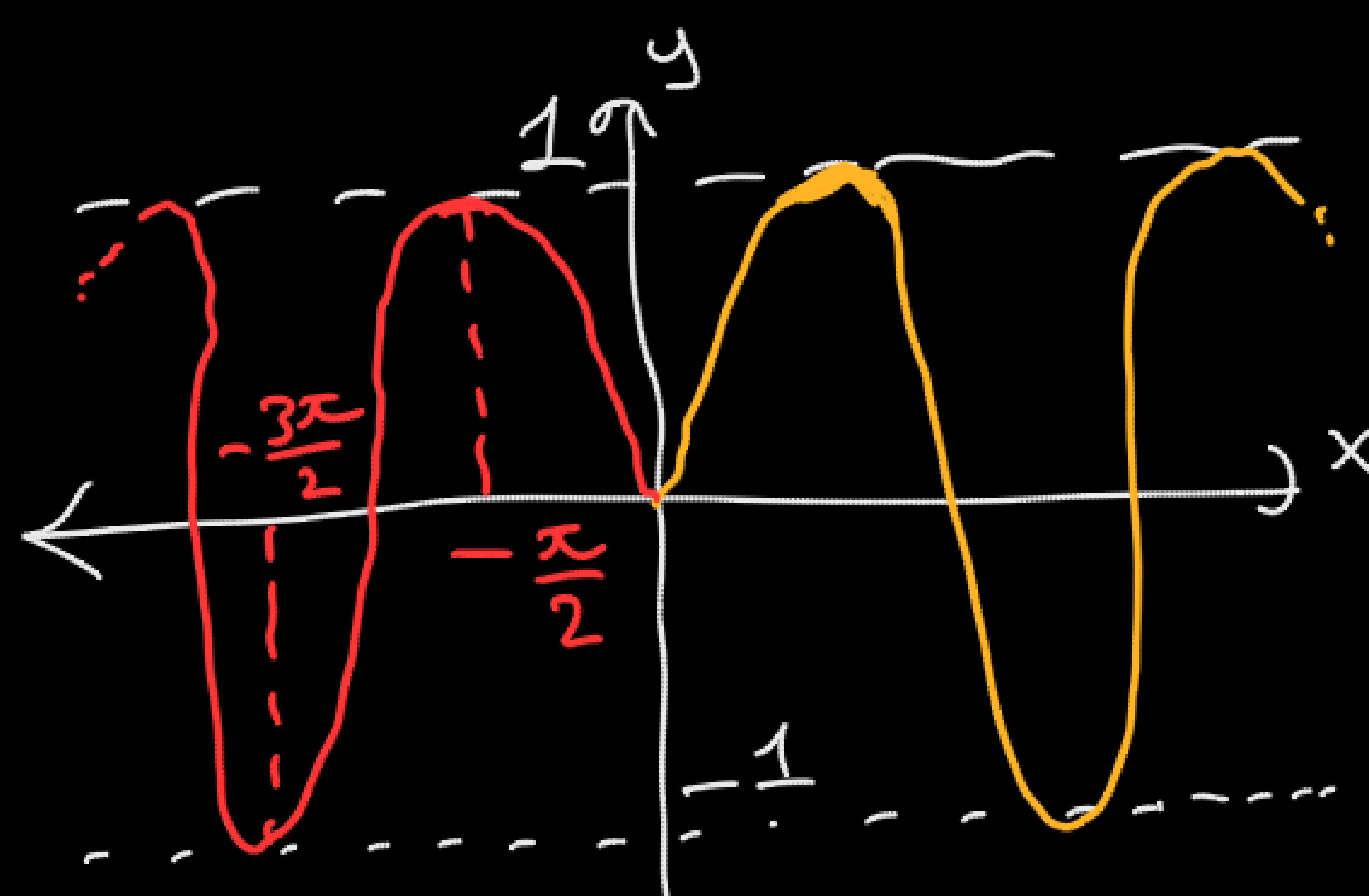
(c) $y=\sqrt{x}$





(b)





(a) It is clear that $y=f(|x|)=f(x)$ for $x\geq 0$.

Observe that if $x<0$, then $y=f(|x|)$ is a reflection of the right hand side of $y=f(x)$ with respect to y-axis (By right hand side of $f(x)$, we mean graph of $f(x)$ with $x\geq 0$)

10) (a) Find (fog) and its domain where $f(x)=x+\frac{1}{x}$ and $g(x)=\frac{x-1}{x+3}$

(b) Given $F(x) = \sin^2(x-5)$, find functions f,g, and h such that $F = f \circ g \circ h$

Soln.:

(a)

1st way: The domain of $(f \circ g)$ consists of those numbers in the domain of g for which $g(x)$ is in the domain of f.

So, $g(x) \neq 0$ since $\text{Dom } f = \mathbb{R} \setminus \{0\}$.

$$g(x) = \frac{x-1}{x+3} \neq 0 \Rightarrow \boxed{x \neq 1}$$

Also, we have $\text{Dom } g = \mathbb{R} \setminus \{-3\}$.

$\therefore \text{Dom}(f \circ g) = \mathbb{R} \setminus \{-3, 1\}$

2nd way: $$(f \circ g)(x) = f(g(x)) = f\left(\frac{x-1}{x+3}\right) = \frac{x-1}{x+3} + \frac{1}{\frac{x-1}{x+3}} = \frac{x-1}{x+3} + \frac{x+3}{x-1}$$

$$= \frac{(x-1)^2 + (x+3)^2}{(x-1)(x+3)}$$

The product $(x-1)(x+3) \neq 0 \Rightarrow x \neq 1,\ x \neq -3$

Hence, $\text{Dom}(f \circ g) = \mathbb{R} \setminus \{-3, 1\}$

(b) $F(x) = [\sin(x-5)]^2$ where $F = f \circ g \circ h$

Choose $h(x) = x - 5$, $g(x) = \sin(x)$ and $f(x) = x^2$.

Then, $(f \circ g \circ h)(x) = f(g(x-5)) = f(\sin(x-5)) = [\sin(x-5)]^2$