MATH 119 2020-1 Recitation Problems for Week 02

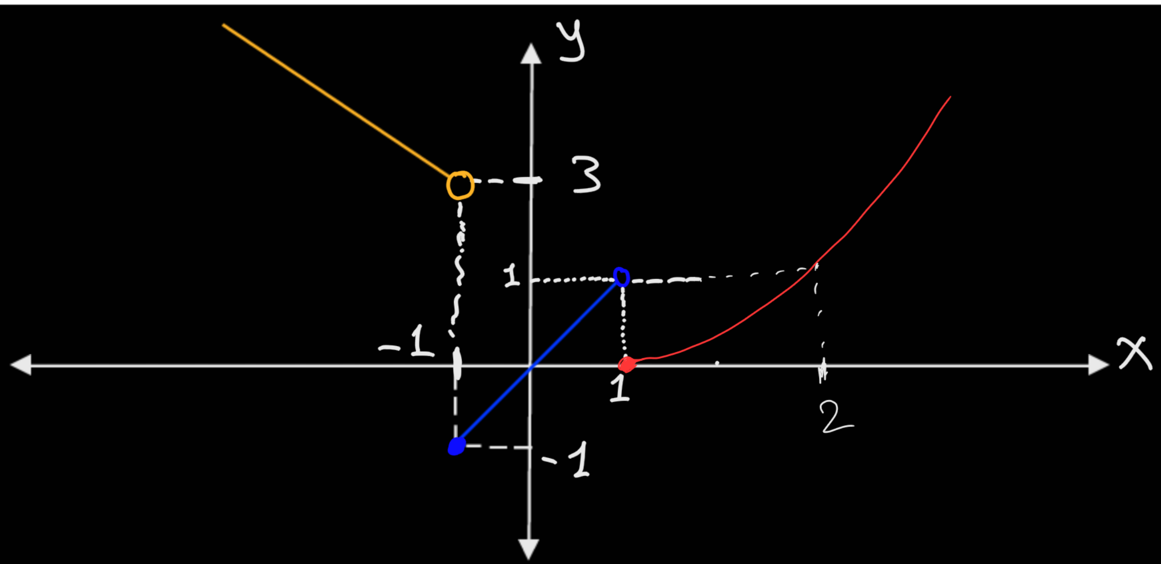
1. Sketch the graph of the following function and evaluate the limits

$$f(x) = \begin{cases} 2-x & \text{if } x < -1 \\ x & \text{if } -1 \leq x < 1 \\ (x-1)^2 & \text{if } x \geq 1 \end{cases}$$

a. $\lim_{x\to -1} f(x) = ?$

b. $\lim_{x\to 0} f(x) = ?$

c. $\lim_{x\to 1} f(x) = ?$



(a) $\lim_{x\to -1} f(x) = ?$

$$\lim_{x\to -1^-} f(x) = \lim_{x\to -1^-} (2-x) = 3$$

$$\Rightarrow 3 \neq -1$$

$$\lim_{x\to -1^+} f(x) = \lim_{x\to -1^+} (x) = -1$$

So, $\lim_{x\to -1} f(x) =$ d.n.e. (does not exist)

(b) $\lim_{x\to 0} f(x) = \lim_{x\to 0} x = 0$

(c) $\lim_{x\to 1^-} f(x) = \lim_{x\to 1^-} x = 1 \neq \lim_{x\to 1^+} f(x) = \lim_{x\to 1^+} (x-1)^2 = 0$

Therefore, $\lim_{x\to 1} f(x) =$ d.n.e.

2. Let $f(x) = \frac{|x-2|}{x-2}$.

a. $\lim_{x\to 0} f(x) = ?$

b. $\lim_{x\to 2} f(x) = ?$

$$f(x) = \begin{cases} +1 & \text{if } x > 2 \\ -1 & \text{if } x < 2 \end{cases}$$

(a) $\lim_{x\to 0} f(x) = \lim_{x\to 0}(-1) = -1$

(b) $\lim_{x\to 2^-} f(x) = -1$, $\lim_{x\to 2^+} f(x) = \lim_{x\to 2^+}(1) = 1$

Since $\lim_{x\to 2^-} f(x) \neq \lim_{x\to 2^+} f(x)$, $\lim_{x\to 2} f(x) = \text{d.n.e.}$

3. Evaluate the limit, if it exists

a. $\lim_{x\to 1} \frac{x^3-1}{x^2-1}$

b. $\lim_{h\to 0} \frac{\sqrt{1+h}-1}{h}$

c. $\lim_{t\to 0} \frac{(3+t)^{-1}-3^{-1}}{t}$

d. $\lim_{x\to 1} \frac{\sqrt{x}-x^2}{1-\sqrt{x}}$

(a) $\lim_{x\to 1} \frac{x^3-1}{x^2-1} = \lim_{x\to 1} \frac{\cancel{(x-1)}(x^2+x+1)}{\cancel{(x-1)}(x+1)} = \lim_{x\to 1} \frac{x^2+x+1}{x+1} = \frac{3}{2}$

(b) $\lim_{h\to 0} \frac{\sqrt{1+h}-1}{h} = \lim_{h\to 0} \frac{(\sqrt{1+h}-1)(\sqrt{1+h}+1)}{h(\sqrt{1+h}+1)} = \lim_{h\to 0} \frac{\cancel{1}+h-\cancel{1}}{h(\sqrt{1+h}+1)} =$

$= \lim_{h\to 0} \frac{\cancel{h}\,1}{\cancel{h}(\sqrt{1+h}+1)} = \frac{1}{2}$ //

(c) $\lim_{t\to 0} \frac{\frac{1}{3+t}-\frac{1}{3}}{t} = \lim_{t\to 0} \frac{\frac{\cancel{3}-\cancel{3}-t}{3(3+t)}}{t} = \lim_{t\to 0} \frac{-1\,\cancel{-t}}{3(3+t)\cdot\cancel{t}}$

$$= \lim_{t\to 0} \frac{-1}{3(3+t)} = \frac{-1}{9}$$

(d) $$\lim_{x\to 1} \frac{\sqrt{x}-(\sqrt{x})^4}{1-\sqrt{x}} = \lim_{x\to 1} \frac{\sqrt{x}\left[1-(\sqrt{x})^3\right]}{1-\sqrt{x}} = \lim_{x\to 1} \frac{\sqrt{x}\cancel{(1-\sqrt{x})}(1+\sqrt{x}+x)}{\cancel{(1-\sqrt{x})}} =$$

$$= \lim_{x\to 1}\left(\sqrt{x}(1+\sqrt{x}+x)\right) = 3$$

4. Let $f(x) = x - \lfloor x \rfloor$

   a. Sketch the graph of f.

   b. If n is an integer, evaluate

      (i) $\lim_{x\to n^-} f(x)$

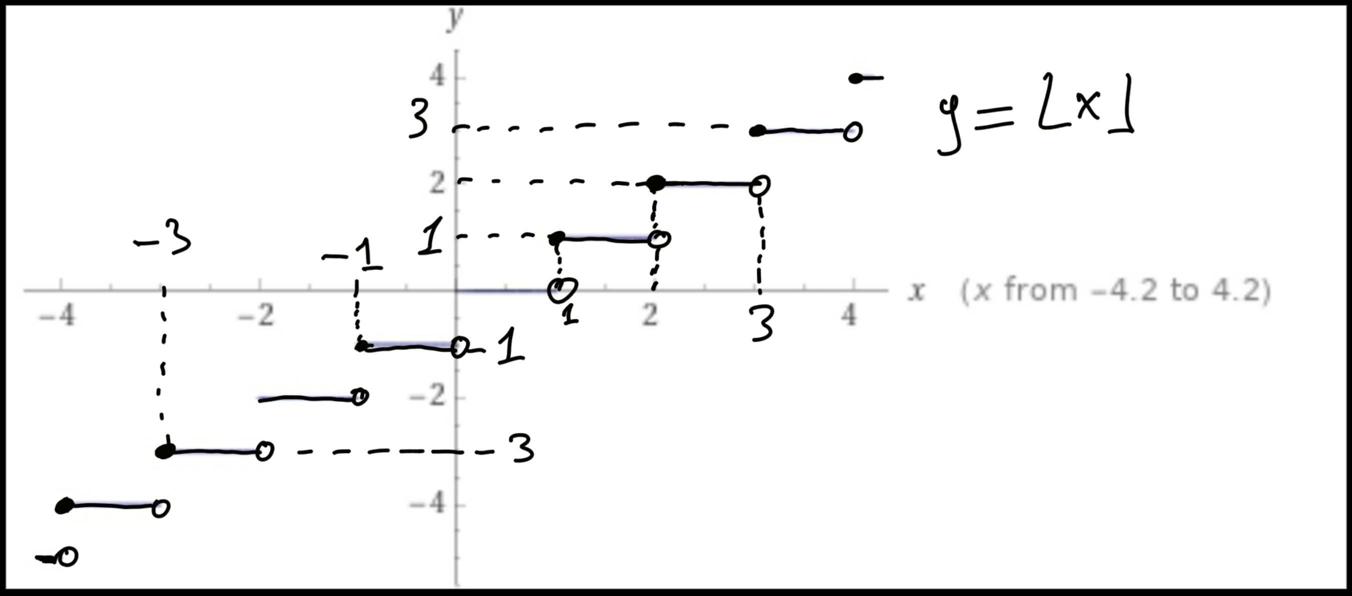
      (ii) $\lim_{x\to n^+} f(x)$

   c. For what values of a does $\lim_{x\to a} f(x)$ exist?

Recall: $y = f(x) = \lfloor x \rfloor = \max\{m \in \mathbb{Z} \mid m \leq x\}$ (The Greatest Integer func.)

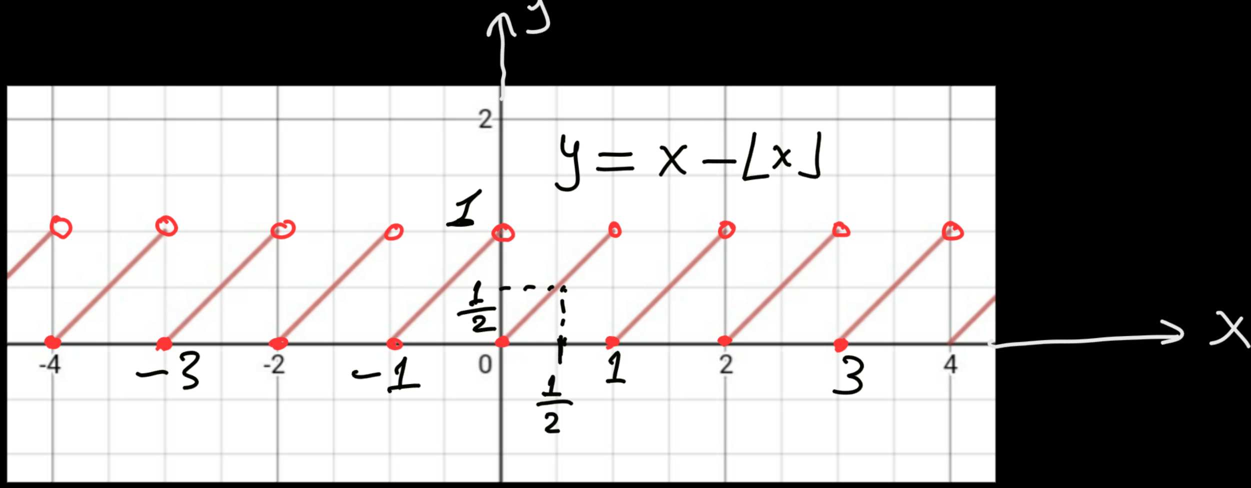
$\lfloor x \rfloor$ = the largest integer less than or equal to x.

ex) $\lfloor 2.7 \rfloor = 2$, $\lfloor 3.5 \rfloor = 3$, $\lfloor 1 \rfloor = 1$, $\lfloor -0.7 \rfloor = -1$, $\lfloor -3 \rfloor = -3$



Sometimes $\lfloor x \rfloor$ is called floor(x)

(a) Therefore, we have



(b) From the graph above, we obtain (Given $n \in \mathbb{Z}$)

(i) $\lim_{x\to n^-} f(x) = 1$, (ii) $\lim_{x\to n^+} f(x) = 0$

(c) From part (b), we see that $\lim_{x\to n} f(x) =$ d.n.e. for any $n \in \mathbb{Z}$.

So, $\lim_{x\to a} f(x) = f(a) = a$ for any $a \notin \mathbb{Z}$

5. If $f(x) = \lfloor x \rfloor + \lfloor -x \rfloor$. Show that $\lim_{x\to 2} f(x)$ exist but is not equal to $f(2)$.

Soln.: If $x \in (n, n+1)$, then $\lfloor x \rfloor = n$ & $\lfloor -x \rfloor = -n-1$

Therefore, $\lim_{x\to 2^+} (\lfloor x \rfloor + \lfloor -x \rfloor) = (2) + (-3) = -1$

$\lim_{x\to 2^-} (\lfloor x \rfloor + \lfloor -x \rfloor) = (1) + (-2) = -1$

Since $\lim_{x\to 2^-} f(x) = \lim_{x\to 2^+} f(x) = -1$, $\boxed{\lim_{x\to 2} f(x) = -1}$

However, $f(2) = \lfloor 2 \rfloor + \lfloor -2 \rfloor = (2) + (-2) = 0 \neq -1$

6. Evaluate

$$I = \lim_{x\to 2} \frac{\sqrt{6-x}-2}{\sqrt{3-x}-1}$$

Soln.: Multiply & divide by conjugates $(\sqrt{6-x}+2)$ and $(\sqrt{3-x}+1)$

So, $I = \lim_{x\to 2} \frac{\cancel{(6-x-4)}(\sqrt{3-x}+1)}{\cancel{(3-x-1)}(\sqrt{6-x}+2)} = \lim_{x\to 2} \frac{\sqrt{3-x}+1}{\sqrt{6-x}+2} = \frac{2}{4}$

7. Prove that

a. $\lim_{x\to 0} x^4 \cos(2/x) = 0$
b. $\lim_{h\to 0^+} \sqrt{h} e^{\sin(\pi/h)} = 0$

(a) $-1 \le \cos\left(\frac{2}{x}\right) \le 1 \Rightarrow -x^4 \le x^4 \cos\left(\frac{2}{x}\right) \le x^4$

Since $\lim_{x\to 0}(-x^4) = 0 = \lim_{x\to 0}(x^4)$, we obtain by Squeeze Theorem

$$\lim_{x\to 0} x^4 \cos\left(\frac{2}{x}\right) = 0$$

(b) $-1 \le \sin\left(\frac{\pi}{h}\right) \le 1 \Rightarrow e^{-1} \le e^{\sin\left(\frac{\pi}{h}\right)} \le e^{1} \Rightarrow$

$\Rightarrow \sqrt{h}\, e^{-1} \le \sqrt{h}\, e^{\sin\left(\frac{\pi}{h}\right)} \le \sqrt{h}\, e^{1}$

$\lim_{h\to 0}\left(\sqrt{h}\, e^{-1}\right) = 0 = \lim_{h\to 0}\left(\sqrt{h}\, e\right)$

$\therefore$ We have by Squeeze Thm, $\lim_{h\to 0} \sqrt{h}\, e^{\sin\left(\frac{\pi}{h}\right)} = 0$

8. Evaluate

$$I = \lim_{x\to\pm\infty} \frac{1}{\sqrt{x^2-2x}-2}$$

$$I = \lim_{x\to\pm\infty} \frac{1}{\sqrt{x^2\left(1-\frac{2}{x}\right)}-2} = \lim_{x\to\pm\infty} \frac{1}{|x|\sqrt{1-\frac{2}{x}}-2}$$

$$\text{So, } \underline{\underline{\lim_{x\to+\infty}}} \frac{1}{\underline{\underline{x}}\left(\sqrt{1-\underbrace{\frac{2}{x}}_{\to 0}}-\underbrace{\frac{2}{x}}_{\to 0}\right)} = 0 = \underline{\underline{\lim_{x\to-\infty}}} \frac{1}{\underline{\underline{-x}}\left(\sqrt{1-\underbrace{\frac{2}{x}}_{\to 0}}-\underbrace{\frac{2}{x}}_{\to 0}\right)}$$

as $x\to\infty$

9. Evaluate

$$\lim_{x\to\infty}\left(\frac{x^2}{x+1}-\frac{x^2}{x-1}\right)$$

$$\lim_{x\to\infty}\left(\frac{x^2(\cancel{x}-1-\cancel{x}-1)}{x^2-1}\right) = \lim_{x\to\infty}\frac{\cancel{x^2}\cdot(-2)}{\cancel{x^2}\cdot\left(1-\frac{1}{x^2}\right)} =$$

$$= \lim_{x\to\infty}\frac{-2}{1-\underbrace{\frac{1}{x^2}}_{\to 0}} = -2$$

as $x\to\infty$