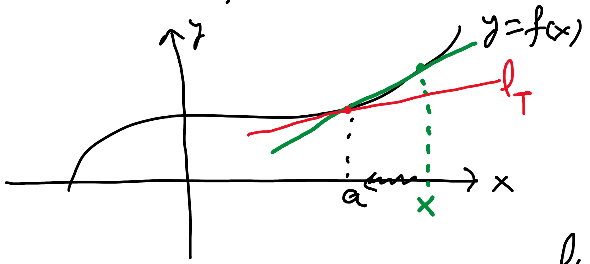
Week 7

Friday, December 3, 2021 1:17 PM

Q1| Find the derivative of the following funcs. by using the definition of the derivative!



$$\lim_{x\to a} \frac{f(x)-f(a)}{x-a} =: f'(a) = \underbrace{m}(l_T)$$

slope of the tang. line of $y=f(x)$ at $x=a$

// $h=x-a$

$$\lim_{h\to 0} \frac{f(a+h)-f(a)}{h}$$

a) $f(x) = 2x^2+3x+6$

$$f'(a) = \lim_{x\to a} \frac{\overbrace{2x^2+3x+6}^{f(x)} - (\overbrace{2a^2+3a+6}^{f(a)})}{x-a} = \lim_{x\to a} \frac{2(x^2-a^2)+3(x-a)}{x-a}$$

$$= \lim_{x\to a} \frac{\cancel{(x-a)}(2(x+a)+3)}{\cancel{x-a}} = 4a+3. \quad \therefore f'(x) = 4x+3$$

b) $f(x) = \dfrac{x}{x+1}$

$$f'(a) = \lim_{x\to a} \frac{f(x)-f(a)}{x-a} = \lim_{x\to a} \frac{\frac{x}{x+1} - \frac{a}{a+1}}{x-a} = \lim_{x\to a} \frac{\frac{x(a+1)-a(x+1)}{(x+1)(a+1)}}{x-a}$$

$$= \lim_{x\to a} \frac{x+\cancel{ax}-\cancel{ax}-a}{(x+1)(a+1)(x-a)} = \lim_{x\to a} \frac{\cancel{x-a}}{\cancel{(x-a)}(x+1)(a+1)} = \frac{1}{(a+1)^2}$$

$$\therefore f'(x) = \frac{1}{(x+1)^2}.$$

Q2| Evaluate the following limits by interpreting each of them as a derivative:

a) $$\lim_{x\to 0} \frac{\cos x - 1}{x} \overset{\left(\frac{0}{0}\right)}{=} \lim_{x\to 0} \frac{f(x)-f(0)}{x-0} = f'(0) = -\sin x\Big|_{x=0} = 0$$

Consider $f(x) = \cos x$

$f(0) = 1$

b) $$\lim_{x\to 2} \frac{\sqrt{4x+1} - 3}{x-2} \overset{\left(\frac{0}{0}\right)}{=} \lim_{x\to 2} \frac{f(x)-f(2)}{x-2} = f'(2) \underset{(*)}{=} \frac{2}{\sqrt{4x+1}}\Bigg|_{x=2} = \underline{\underline{2/3}}$$

Say $f(x) = \sqrt{4x+1}$

$f(2) = 3$

$\left[f'(x) = (\sqrt{4x+1})' = ((4x+1)^{1/2})' = \frac{1}{2} \cdot (4x+1)^{-1/2} \cdot 4 \right]$

c) $\lim_{x\to 0} \frac{(\cos x - 1)\sin x}{x^2} = \lim_{x\to 0} \left(\frac{\cos x - 1}{x} \cdot \frac{\sin x}{x} \right) \quad (*)$

$\lim_{x\to 0} \frac{\cos x - 1}{x - 0} = \lim_{x\to 0} \frac{f(x)-f(0)}{x-0} = f'(0) = -\sin x \Big|_{x=0} = 0 \quad \therefore$ It exists.

say $f(x) = \cos x$
$f(0) = 1$

$\lim_{x\to 0} \frac{\sin x}{x} = \lim_{x\to 0} \frac{g(x)-g(0)}{x-0} = g'(0) = \cos x \Big|_{x=0} = 1$ So, it exists.

say $g(x) = \sin x$
$g(0) = 0$

As both limits exist, $(*) = \lim_{x\to 0} \frac{\cos x - 1}{x-0} \cdot \lim_{x\to 0} \frac{\sin x}{x} = 0 \cdot 1 = 0$

d) $\lim_{x\to 1} \frac{x^a - 1}{x - 1}$ exc! $a \in \mathbb{R}$

Q3) Show that f is constant on $\mathbb{R}$ if $|f(x) - f(y)| \leq (x-y)^2$, $\forall x,y \in \mathbb{R}$

$f(x) \leq g(x)$ & they are diff'ble $\not\Rightarrow$ $f'(x) \leq g'(x)$

$f(x) = -\frac{1}{x}$ > on $(0,\infty)$ But $f'(x) = \frac{1}{x^2} \not\leq 0$
$g(x) = 153$

Given any $a \in D_f$,

$$0 \leq |f(x) - f(a)| \leq (x-a)^2$$

$$\underset{x \neq a}{\Longrightarrow} \quad 0 \leq \left| \frac{f(x)-f(a)}{x-a} \right| \leq \left| \frac{(x-a)^2}{x-a} \right| = |x-a|$$

as $x \to a$: 0

By Squeeze thm, $\lim_{x\to a} |x-a| = 0 \Rightarrow \lim_{x\to a} \left| \frac{f(x)-f(a)}{x-a} \right| = 0$

Then, $\lim_{x\to a} \frac{f(x)-f(a)}{x-a} = 0 := f'(a)$.

So, $f'(a) = 0$ for all $a \in \mathbb{R}$. So, f is constant on its ~~domain~~.

Remark: $f'(x) = 0 \not\Rightarrow f$ is constant.

$$f(x) = \begin{cases} 1 & x \geq 0 \\ 2 & x < 0 \end{cases} \Rightarrow f'(x) = 0 \text{ on } \mathbb{R} - \{0\}$$

exc!

But, f is not constant

Q4) Let f be defined as $f(x) = \begin{cases} x^a \sin\left(\frac{1}{x}\right) & \text{if } x \neq 0 \\ 0 & \text{if } x = 0 \end{cases}$

a) Determine $a \in \mathbb{R}$ s.t. f is diff'ble at 0.

$$f'(0) = \lim_{\substack{x \to 0 \\ x \neq 0}} \frac{f(x) - f(0)}{x - 0} = \lim_{x \to 0} \frac{x^a \cdot \sin\left(\frac{1}{x}\right)}{x} = \lim_{x \to 0} x^{a-1} \cdot \sin\left(\frac{1}{x}\right)$$

If $a = 1$, $\lim_{x \to 0} \sin\left(\frac{1}{x}\right) = $ d.n.e.

If $a > 1 \implies 0 \leq \left| x^{a-1} \cdot \sin\left(\frac{1}{x}\right) \right| \leq \left| x^{a-1} \right|$. By Squeeze thm,

as $x \to 0$

$$\lim_{x \to 0} |x^{a-1}| = \lim_{x \to 0} 0 = 0 \implies \lim_{x \to 0} \left| x^{a-1} \cdot \sin\left(\frac{1}{x}\right) \right| = 0$$

So, $\lim_{x \to 0} x^{a-1} \cdot \sin\left(\frac{1}{x}\right) = 0$

If $a < 1$, $\lim_{x \to 0} x^{a-1} \sin\left(\frac{1}{x}\right) = \lim_{x \to 0} \frac{\sin\left(\frac{1}{x}\right)}{x^{1-a}} \underset{(*)}{=}$ d.n.e.

(*) $\lim_{x \to 0} \frac{\sin\left(\frac{1}{x}\right)}{x^{1-a}} = L \in \mathbb{R}$. , $\lim_{x \to 0} x^{1-a} = 0$ ($1 - a > 0$)

$$\lim_{x \to 0} \left(\underbrace{\frac{\sin(1/x)}{x^{1-a}}}_{\to L} \cdot \underbrace{x^{1-a}}_{\to 0} \right) = \text{↯}.$$

$$\parallel$$

$$\lim_{x \to 0} \sin\left(\frac{1}{x}\right)$$