Week 5

Friday, November 19, 2021 1:35 PM

Q1) If $\lim_{x\to 0} \frac{f(x)}{x^2} = -2$, find $\lim_{x\to 0} f(x)$ and $\lim_{x\to 0} \frac{f(x)}{x}$.

$$\lim_{x\to 0} f(x) = \lim_{x\to 0} \frac{f(x)}{x^2}\cdot x^2 \overset{\text{as both limits exist!}}{=} \underbrace{\lim_{x\to 0}\frac{f(x)}{x^2}}_{-2}\cdot\underbrace{\lim_{x\to 0} x^2}_{=0} = -2\cdot 0 = 0$$

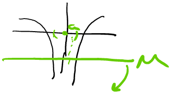
$$\lim_{x\to 0} \frac{f(x)}{x} = \lim_{x\to 0} \frac{f(x)}{x^2}\cdot x \overset{\text{as both limits exist.}}{=} \lim_{x\to 0}\frac{f(x)}{x^2}\cdot\lim_{x\to 0} x = -2\cdot 0 = 0$$

$\left(\lim_{x\to 0} \frac{f(x)}{x^2} = -2,\ \lim_{x\to 0} x = 0 \right)$

Q2) Prove the followings:

a) $\lim_{x\to 1^{+}} \frac{1}{1-x} = -\infty$

Recall: $\lim_{x\to a} f(x) = -\infty \iff \forall M<0,\ \exists \delta>0$ s.t. $0<|x-a|<\delta \Rightarrow f(x)<M$



Given any $M<0$. Choose $\delta = -\frac{1}{M} > 0$

due to right limit

If $0 < x-1 < \delta$,

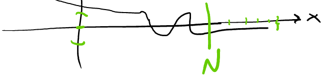
$$f(x) = \frac{1}{1-x} = \frac{-1}{x-1} < \frac{-1}{\delta} = M \text{ as } \delta = -\frac{1}{M} > 0$$

$\boxed{<M}$

$\frac{1}{x-1} > \frac{1}{\delta} \Rightarrow \frac{-1}{x-1} < \frac{-1}{\delta}$ by assumption

b) $\lim_{x\to\infty} \frac{1}{(x+1)^2} = 0$

Recall: $\lim_{x\to\infty} f(x) = L^{\in\mathbb{R}} \iff \forall \varepsilon>0,\ \exists N>0$ s.t. $x>N \Rightarrow |f(x)|<\varepsilon$



Given $\varepsilon > 0$. Choose $N = \dfrac{1}{\sqrt{\varepsilon}} \ldots > 0$

If $x > N$, $|f(x)| = \left|\dfrac{1}{(x+1)^2}\right| = \dfrac{1}{x^2 + \underbrace{(2x+1)}_{>0 \text{ for } x > N > 0}} \leq \dfrac{1}{x^2} \overset{\text{by assumption}}{<} \dfrac{1}{N^2} = \varepsilon$ as $N = \dfrac{1}{\sqrt{\varepsilon}}$ $\boxed{< \varepsilon}$

c) $\lim\limits_{x \to \infty} \sqrt{x} = \infty \iff \forall M > 0, \exists N > 0$ s.t. $x > N \Rightarrow \sqrt{x} > M$

Given $M > 0$. Choose $N = M^2 \ldots > 0$

If $x > N$, $\sqrt{x} > \sqrt{N} = M$ as $N = M^2$ $\boxed{> M}$

Q3) Find following limits:

a) $\lim\limits_{x\to\infty} \dfrac{x + \sin^3 x}{5x+6} \overset{[\frac{\infty}{\infty}]}{=} \lim\limits_{x\to\infty} \dfrac{\cancel{x}\left(1 + \cancel{\frac{\sin^3 x}{x}}^{\,0\ (*)}\right)}{\cancel{x}\left(5 + \cancel{\frac{6}{x}}^{\,0}\right)} = \dfrac{1}{5}$

(*) $0 \leq \left|\dfrac{\sin^3 x}{x}\right| \leq \dfrac{1}{|x|}$ as $x \to \infty$ → 0

By Squeeze thm, $\lim\limits_{x\to\infty} \left|\dfrac{\sin^3 x}{x}\right| = 0$

Then, $\lim\limits_{x\to\infty} \dfrac{\sin^3 x}{x} = 0$

b) $\lim\limits_{x\to 0} \dfrac{\tan^2 x + \sin(x^3)}{2x^2} = \lim\limits_{x\to 0}\left(\dfrac{\tan^2 x}{2x^2} + \dfrac{\sin(x^3)}{2x^2}\right)$ $\boxed{\lim\limits_{x\to 0} \dfrac{\sin x}{x} = 1}$ $\boxed{\lim\limits_{x\to 0} \dfrac{\tan x}{x} = 1}$

$= \lim\limits_{x\to 0}\left(\left(\cancel{\dfrac{\tan x}{x}}^{\,1}\right)^2 \cdot \dfrac{1}{2} + \cancel{\dfrac{\sin(x^3)}{2x^3}}^{\,1} \cdot \cancel{x}^{\,0}\right)$

$= \dfrac{1}{2} + 0 = \dfrac{1}{2}$

Remark: ~~$\lim\limits_{x\to 0}\left(\dfrac{\sin x}{x}\right) = \lim\limits_{x\to 0}\left(\dfrac{0}{x}\right) = \lim\limits_{x\to 0} 0 = 0$~~ Not possible.

c) $\lim\limits_{x\to -\infty}\left(2x - \sqrt{4x^2 - 3x}\right)$

1st way

$= \lim\limits_{x\to -\infty} \dfrac{\left(2x - \sqrt{4x^2-3x}\right)\left(2x + \sqrt{4x^2-3x}\right)}{\left(2x + \sqrt{4x^2-3x}\right)} = \lim\limits_{x\to -\infty} \dfrac{4x^2 - (4x^2 - 3x)}{2x + \sqrt{4x^2-3x}}$

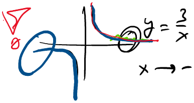
$$= \lim_{x\to-\infty} \frac{3x}{2x + \underbrace{|x|}_{-x}\sqrt{4-\frac{3}{x}}} = \lim_{x\to-\infty} \frac{3\cancel{x}}{\cancel{x}\left(2-\sqrt{4-\frac{3}{x}}\right)}$$

$\sqrt{4-\frac{3}{x}} \to 2$ from right

$\frac{3}{x} \to 0$ from left

$\left(2-\sqrt{4-\frac{3}{x}}\right) \to 0$ from left

$$= -\infty$$



$x\to-\infty \Rightarrow \frac{3}{x} \longrightarrow 0$ from right

$$\lim_{x\to-\infty} \frac{3}{x} = 0$$

$\lim_{x\to0} \frac{1}{x}$ = dne.

$\lim_{x\to0^+} \frac{1}{x} = +\infty$ $\qquad$ $\lim_{x\to0^-} \frac{1}{x} = -\infty$

Then, $\lim_{x\to-\infty} \frac{1}{\frac{1}{x}} = \lim_{x\to-\infty} x = -\infty$ ✓

$\frac{1}{x} \to 0^-$

↳ taking negative values.

2nd way

$$\lim_{x\to-\infty} \left(\underset{-\infty}{2x} - \underset{+\infty}{\sqrt{4x^2-3x}}\right) = -\infty$$

$$= \lim_{x\to-\infty}\left(2x - \underbrace{|x|}_{-x}\sqrt{4-\frac{3}{x}}\right) = \lim_{x\to-\infty} x\left(2+\sqrt{4-\frac{3}{x}}\right) = -\infty$$

$\sqrt{4-\frac{3}{x}} \to 4$ $\qquad$ $4-\frac{3}{x} \to 4$

∴ Its limit dne.

Do not write!

~~$-\infty \cdot 4 = -\infty$~~

~~$-\infty - \infty = -\infty$~~

$\pi + e = \checkmark \in \mathbb{R}$

d) $\lim_{x\to\infty}(x+\cos x) = \lim_{x\to\infty} x\left(1+\frac{\cos x}{x}\right) = \infty$

$\frac{\cos x}{x} \to 0$ (*) $\qquad$ $\left(1+\frac{\cos x}{x}\right) \to 1$

(*) $0 \le \left|\frac{\cos x}{x}\right| \le \frac{1}{|x|}$

as $x\to+\infty$, $\to 0$

By Squeeze thm,

$$\lim_{x\to\infty}\left|\frac{\cos x}{x}\right| = 0 \Rightarrow \lim_{x\to\infty} \frac{\cos x}{x} = 0$$

OR

$$-1 \le \cos x \le 1 \Rightarrow -\frac{1}{x} \le \frac{\cos x}{x} \le \frac{1}{x} \quad \text{for } x>0$$

as $x\to+\infty$, $\to 0$ $\qquad$ ✓ as $x\to+\infty$

$$\frac{1}{x} \le \frac{\cos x}{x} \le -\frac{1}{x} \quad \text{for } x<0$$

No need!

e) $\lim_{x\to2} \frac{\sqrt{4-4x+x^2}}{x-2} \left[\frac{0}{0}\right] = \lim_{x\to2} \frac{\sqrt{(x-2)^2}}{x-2} = \lim_{x\to2} \frac{|x-2|}{x-2}$

$$\left.\begin{array}{l}\lim\limits_{x\to 2^+} \dfrac{x-2}{x-2} = 1 \\ \text{but} \\ \lim\limits_{x\to 2^-} \dfrac{-\cancel{(x-2)}}{\cancel{x-2}} = -1\end{array}\right\} \neq \quad \text{So its limit d.n.e.}$$

f) $$\lim_{x\to 8} \frac{x^{2/3}-4}{x^{1/3}-2} \overset{\left[\frac{0}{0}\right]}{=} \lim_{x\to 8} \frac{\cancel{(x^{1/3}-2)}\cdot(x^{1/3}+2)}{\cancel{x^{1/3}-2}} = \lim_{x\to 8}(x^{1/3}+2) = 4$$

g) $$\lim_{x\to 0} \frac{\sqrt{x^2+25}-5}{x^2} \overset{\left[\frac{0}{0}\right]}{=} \underline{\underline{\text{exc.}}}$$

Q4) a) Prove that if $\lim\limits_{x\to 0} \dfrac{f(x)}{x} = L$ for some $L \in \mathbb{R}$ and $b \neq 0$, then $\lim\limits_{x\to 0} \dfrac{f(b.x)}{x} = b.L$

$$\lim_{x\to 0} \frac{f(b.x)}{x} \underset{\substack{u=bx,\ b\neq 0\\ u\to 0 \text{ as } x\to 0}}{=} \lim_{u\to 0} \frac{f(u)}{u/b} = \lim_{u\to 0} b.\frac{f(u)}{u} = b.\lim_{u\to 0} \frac{f(u)}{u} = b.L.$$

b) what happens if $b=0$?

$$\lim_{x\to 0} \frac{f(b.x)}{x} = \lim_{x\to 0} \frac{f(0)}{x} \xrightarrow{\text{Constant}} = \begin{cases} 0 & \text{if } f(0)=0 \\ f(0).\lim\limits_{x\to 0} \dfrac{1}{x} = \underline{\text{d.n.e}} & \text{if } f(0)\neq 0 \end{cases}$$

c) By using part (a), find $\lim\limits_{x\to 0} \dfrac{\sin(2x)}{x}$.

$$= \lim_{u\to 0} \frac{\sin(u)}{u/2} = 2\lim_{u\to 0} \frac{\sin u}{u} = 2.$$

$u = 2x$

$u\to 0$ as $x \to 0$

substitution

! $\lim\limits_{x\to 0} \dfrac{\sin(2x)}{x} = 2$

do not write directly!!

Q5) Let the function f defined as follows,

$$f(x) = \begin{cases} ax^2+1 & \text{if } x<0 \\ x+b & \text{if } \underline{x\geq 0} \end{cases}$$

Find a & b values which makes $f(x)$ cont. everywhere

At $x=0$:

$\lim_{x\to0^+} f(x) = \lim_{x\to0^-} f(x) \overset{=L}{\Longrightarrow} \lim_{x\to0} f(x) = L$, $L\in\mathbb{R}$

↪ its limit exists. at $x=0$

$\lim_{x\to0} f(x) = f(0)$

↪ f is continuous at $x=0$

$\left.\begin{array}{l} \lim_{x\to0^+} (x+b) = b \\ \lim_{x\to0^-} (ax^2+1) = 1 \end{array}\right) =$ If $b=1$, $\lim_{x\to0} f(x) = 1 \underset{\checkmark}{=} f(0)=1$

$\therefore$ f is cont. at $x=0$

Away from $x=0$,

If $x>0$, ax^2+1 is cont. on $(0,\infty)$ as it is a polynomial.

If $x<0$, $x+1$ is cont on $(-\infty,0)$ as it is a polynomial

$\therefore$ If $b=1$, $a\in\mathbb{R}$, $f(x)$ is cont everywhere.

5)

$$\lim_{x\to\infty}\frac{2^{x^2}+1}{3^{x^2}+1} = \lim_{x\to\infty}\frac{2^{x^2}\left(1+\cancelto{0}{\frac{1}{2^{x^2}}}\right)}{3^{x^2}\left(1+\cancelto{0}{\frac{1}{3^{x^2}}}\right)} ~~= \lim_{x\to\infty}\frac{2^{x^2}}{3^{x^2}} = \lim_{x\to\infty}\left(\frac{2}{3}\right)^{x^2} = 0~~$$

Partial limits.

$$= \lim_{x\to\infty} \cancelto{0}{\left(\frac{2}{3}\right)^{x^2}} \frac{\left(1+\cancelto{0}{\frac{1}{2^{x^2}}}\right)}{\left(1+\cancelto{0}{\frac{1}{3^{x^2}}}\right)} = \frac{0\cdot1}{1} = 0$$