Week 4

Friday, November 12, 2021 1:21 PM

Quiz-1 $f(x) = |x-153| - |x+153| + 6x$

a) $f(x) < 0$

i) If $x < -153$

$f(x) = -(+x-153) - (-(x+153)) + 6x = 306 + 6x < 0$

$\Rightarrow x < -51$. So, $\boxed{x < -153}$

ii) If $-153 \leq x \leq 153$

$f(x) = -(x-153) - (x+153) + 6x = -2x + 6x < 0$

$\Rightarrow x < 0$. So, $\boxed{-153 \leq x < 0}$

iii) If $x > 153$, $\Rightarrow f(x) = -306 + 6x < 0$

$\Rightarrow x < 51$. No soln. in this interval.

$\therefore$ Soln. set is $(-\infty, 0)$

b) For any $x \in \text{Dom} f = \mathbb{R}$,

$f(-x) = \underbrace{|-x-153|}_{|x+153|} - \underbrace{|-x+153|}_{x-153} - 6x = -f(x)$

$\therefore$ f is odd.

c) • $x < 0 \Rightarrow f(x) < 0$ from part a). Then, for $x > 0$,

$f(x) = f(-(-x)) \underset{\text{f is odd from part b)}}{=} -\underbrace{\overset{<0 \text{ as } x>0}{f(-x)}}_{<0} > 0$

Then, $f(x) \neq 0$ for $x \in \mathbb{R} - \{0\}$. But $f(0) = 0$. So, the only value making f 0 is 0.

Q1) Use the formal definition $(\varepsilon - \delta)$ to prove the following:

a) $\lim_{x \to -2} 3x + 1 = -5$

Recall: $\lim_{x \to a} f(x) = L$, $L \in \mathbb{R}$ $\iff$ $\underset{\text{for all}}{\forall} \varepsilon > 0, \underset{\text{there exists}}{\exists}$ a $\delta > 0$ s.t.

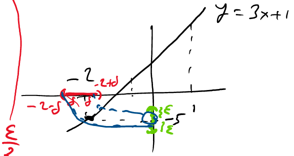
i.e. it exists

if $0 < |x-a| < \delta$, then $|f(x) - L| < \varepsilon$.

Given $\varepsilon > 0$. Choose $\delta = \varepsilon/3 > 0$

If $0 < |x-(-2)| < \delta$, then

$|f(x) - (-5)| = |3x+6| = 3|x+2| \underset{\text{by assumption}}{<} 3 \cdot \delta = \varepsilon$ as $\delta = \frac{\varepsilon}{3}$



b) $\lim_{x \to -1} (x^2 - 5x + 8) = 14$

Given $\varepsilon > 0$. Choose $\delta = \min\{1, \frac{\varepsilon}{8}\} > 0$

If $0 < |x - (-1)| < \delta$, then

$$|x^2 - 5x + 8 - 14| = |x^2 - 5x \underset{-6}{-6}| = |x+1| . |x-6| \underset{\text{by assump.}}{<} \delta . |x-6| \underset{\text{by } (*)}{<} \delta . 8 \underset{\text{as } \delta = \varepsilon/8}{=} \varepsilon \quad \boxed{< \varepsilon}$$

$0 < \delta < 1$ OR $\delta = 1$ ~~OR $\delta > 1$~~ ~ use $\delta = 1$ & it works!!

(*) We can say $\delta \leq 1$. $(0 < |x+1| < \delta)$

$$|x-6| = |x+1-7| \underset{\text{by triangle ineq.}}{\leq} |x+1| + |-7| \leq \underset{\delta}{1} + 7 = 8$$

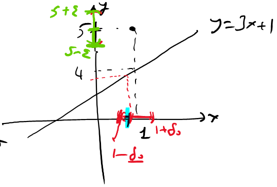
$+(-7)$

c) $\lim_{x \to 1} (3x+1) \neq 5$

Assume $\lim_{x \to 1} 3x+1 = 5$. Then, $\forall \varepsilon > 0$, $\exists \delta > 0$ s.t. if $0 < |x-1| < \delta$

$\Rightarrow |3x+1-5| < \varepsilon$

Choose $\varepsilon = \frac{1}{2} > 0$. Then, we know that there exists a $\delta_0 > 0$ for this ε.



$0 < |x-1| < \delta_0 \Longrightarrow |3x+1-5| < \varepsilon$

$x = 1 - \frac{\delta_0}{2} \Rightarrow x \in (1-\delta_0, 1+\delta_0) - \{1\}$. But

$$|3.(1 - \tfrac{\delta_0}{2}) - 4| = |-\tfrac{3\delta_0}{2} - 1| = 1 + \tfrac{3}{2}\delta_0 \not< 1/2 .$$

We get a contradiction.

d) $\lim_{x \to 0} \sin \frac{1}{x} = $ d.n.e. (Show $\lim_{x \to 0} \sin \frac{1}{x} \neq L \quad \forall L \in \mathbb{R}$)

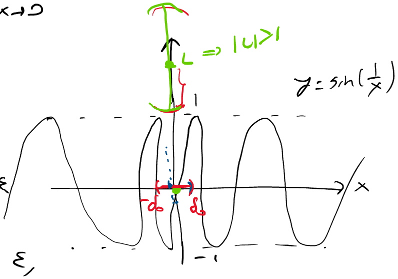
1st Case: For $|L| > 1$,

Assume $\lim_{x \to 0} \sin \frac{1}{x} = L$. Then,

$\forall \varepsilon > 0, \exists \delta > 0$ s.t. $0 < |x| < \delta \Rightarrow |\sin \frac{1}{x} - L| < \varepsilon$

Choose $\varepsilon = |L| - 1 > 0$. Then, for this ε,

there exists a $\delta_0 > 0$ s.t. $0 < |x| < \delta_0 \Rightarrow |f(x) - L| < \varepsilon$



Choose $x = \frac{\delta_0}{2}$. Then, $|\sin(\frac{2}{\delta_0}) - L| \overset{?}{<} |L| - 1 \Rightarrow -|L| + 1 < \sin(\frac{2}{\delta_0}) - L < |L| - 1$

$\Rightarrow \underline{L-|L|}+1 < \sin\frac{2}{\delta_0} < L+|L|-1$ $\nearrow L<0 \Rightarrow \sin(\frac{2}{\delta}) < -1$ ↯
$\searrow L>0 \Rightarrow 1 < \sin(\frac{2}{\delta})$ ↯

So, it cannot happen. If $|L|>1$, $\lim_{x\to 0} \sin(\frac{1}{x}) \neq L$

→ 2nd Case for $|L| \leq 1$, assume $\lim_{x\to 0} \sin(\frac{1}{x}) = L$

$\forall \varepsilon > 0, \exists \delta > 0$ s.t. $0 < |x| < \delta \Rightarrow |\sin(\frac{1}{x}) - L| < \varepsilon$.

Choose $\varepsilon = \frac{1}{2} > 0$. Then $\exists \delta_0 > 0$ s.t.

if $0 < |x| < \delta_0$, then $|f(x) - L| = |\sin(\frac{1}{x}) - L| \overset{?}{<} \varepsilon$

If $0<|x|<\delta_0$, there exists $r_1 = \frac{2}{\pi(1+4n_0)}$, $r_2 = \frac{2}{\pi(3+4n_0)}$ s.t.

$0 < r_2 < r_1 < \delta_0$ for some $n_0 \in \mathbb{N}$.

For $r_1 = \frac{2}{\pi(1+4n_0)}$, $|f(r_{1_0}) - L| = |\underbrace{\sin(\frac{\pi}{2}+2n_0\pi)}_{+1} - L| < 1/2$ (*)

For $r_2 = \frac{2}{\pi(3+4n_0)}$, $|f(r_2) - L| = |\underbrace{\sin(\frac{3\pi}{2}+2n_0\pi)}_{-1} - L| < 1/2$ (**)

$\sin(\overbrace{\frac{\pi}{2}+2n\pi}^{\frac{\pi+4n\pi}{2}=\frac{1}{x}}) = 1$ $\rightarrow x = \frac{2}{\pi(1+4n)}$

$\sin(\underbrace{\frac{3\pi}{2}+2n\pi}_{=\frac{3\pi+4n\pi}{2}=\frac{1}{x}}) = -1$ $\rightarrow x = \frac{2}{\pi(3+4n)}$

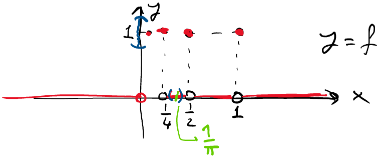
Then, (*) $\Rightarrow -\frac{1}{2} < L < \frac{3}{2}$
exc.
(**) $\Rightarrow -\frac{3}{2} < L < -\frac{1}{2}$

$\Big\rangle$ $\therefore$ No common soln. So, $\lim_{x\to 0} \sin(\frac{1}{x}) \neq L$ for $|L| \leq 1$

□

Q2] Let $f(x) = \begin{cases} 1 & \text{if } x = \frac{1}{n} \text{ for some } n \in \mathbb{N} \\ 0 & \text{o/w} \end{cases}$

a) Show that if $\underline{a \neq 0}$, $\lim_{x\to a} f(x) = \underline{0}$



On $(-\infty, 0)$ and $(1, \infty)$, $f(x)$ behaves as constant function.

So, $\lim_{x\to a} f(x) = 0$ for $a \in (-\infty, 0)$ & $(1, \infty)$ (exc: Prove it!)

For $\underset{x_0}{a} \neq \frac{1}{n}$ for some $n \in \mathbb{N}$ (i.e. $x = \frac{1}{\pi}$)

$\lim_{x\to a} f(x) = 0$

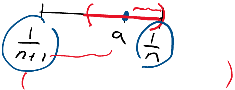
Given $\varepsilon > 0$. Choose $\delta = \min\{\frac{1}{n} - a, a - \frac{1}{n+1}\} > 0$

If $0 < |x-a| < \delta$, then $|f(x) - 0| = |f(x)| = \underset{0 \text{ or } 1}{0} < \varepsilon$ □

finding x making f(x) 0.

$(\varepsilon, \varepsilon)$

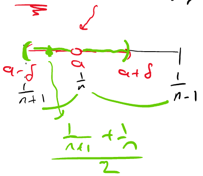
As $a \neq \frac{1}{n}$, $a \in (\frac{1}{n+1}, \frac{1}{n})$ for some $n \in \mathbb{N}$.

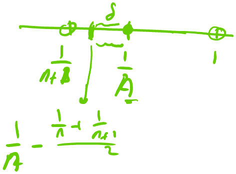


For $a = \frac{1}{n}$, $n \in \mathbb{N}$

Given $\varepsilon > 0$. Choose $\delta = \frac{1}{n} - \frac{\frac{1}{n} + \frac{1}{n+1}}{2} > 0$

If $0 < |x-a| < \delta$, $|f(x) - 0| = |f(x)| = 0 < \varepsilon$





b) Show that $\lim_{x\to 0} f(x)$ d.n.e.

$\lim_{x\to 0^-} f(x) = \lim_{x\to 0^-} 0 = 0$. Enough to show that $\lim_{x\to 0^+} f(x) \neq 0$

Assume $\lim_{x\to 0^+} f(x) = 0$

$\forall \varepsilon > 0, \exists \delta > 0$ s.t. if $0 < x < \delta$, $|f(x) - 0| = |f(x)| < \varepsilon$

Choose $\varepsilon = 1/3 > 0$

$0 < x < \delta \implies |f(x)| < \varepsilon = 1/3$

we can find an irrational number r in $(0, \delta)$ for any $\delta > 0$

So, $|f(r)| = 1 \not< 1/3$ ↯ Contradiction.

Q3) What is the domain of $x^3 \cos(\frac{153}{x})$? Use the formal def'n to prove that $\lim_{x\to 0} x^3 \cos(\frac{153}{x}) = 0$

. Domain $= \mathbb{R} - \{0\}$

Given any $\varepsilon > 0$. Choose $\delta = \sqrt[3]{\varepsilon} \ldots > 0$.

as $\delta = \sqrt[3]{\varepsilon}$

If $\underbrace{0 < |x| < \delta}_{x \in (-\delta,\delta) - \{0\}}$, $\left| x^3 \cos\left(\frac{153}{x}\right) - 0 \right| = |x|^3 \cdot \left|\cos\left(\frac{153}{x}\right)\right| \leq |x|^3 < \delta^3 = \varepsilon$ $\boxed{<\varepsilon}$

as $\left|\cos\left(\frac{153}{x}\right)\right| \leq 1$

by assump.

Q4) Let $\lim_{x\to 1} f(x) = 10$ & $\lim_{x \to 1} g(x) = 20$, find

a) $\lim_{x\to 1} (f(x))^2 = \lim_{x\to 1} f(x) \cdot \lim_{x \to 1} f(x) = 10 \cdot 10 = 100$

as $\lim_{x\to 1} f(x)$ exists.

b) $\lim_{x\to 1} (g(x) - 10) = \lim_{x\to 1} g(x) - \lim_{x \to 1} 10 = 20 - 10 = 10$

as both limits exists

a constant func.

Q5) Determine whether the following statements are true or false.

a) If $\lim_{x\to 0} f^2(x) = 9$, then $\left(\lim_{x\to 0} f(x)\right)^2 = 9$

False: Consider $f(x) = \frac{3|x|}{x}$ & $\lim_{x\to 0} f^2(x) = \lim_{x\to 0} \frac{9\cancel{x^2}}{\cancel{x^2}} = 9$

But, $\lim_{x\to 0} \frac{|x|}{x}$ d.n.e.

b) $\lim_{x\to a} f(x)$ exists & $\lim_{x\to a} g(x)$ dne $\Rightarrow \lim_{x\to a} (f(x)+g(x))$ dne

True: Assume $\lim_{x\to a}(f(x)+g(x))$ exists. As $\lim_{x\to a} f(x)$ exists,

$\lim_{x\to a}(f(x)+g(x)) - \lim_{x\to a} f(x) = \lim_{x\to a}(\cancel{f(x)} + g(x) - \cancel{f(x)})$ exists,

So, $\lim_{x\to a} g(x)$ exists. ↯ It contradicts to non-existence of $\lim_{x\to a} g(x)$ □