Week 6

Thursday, November 25, 2021 11:21 AM

Q1) Determine where the following fncs are continuous.

a) $f(x) = \cos x^2$

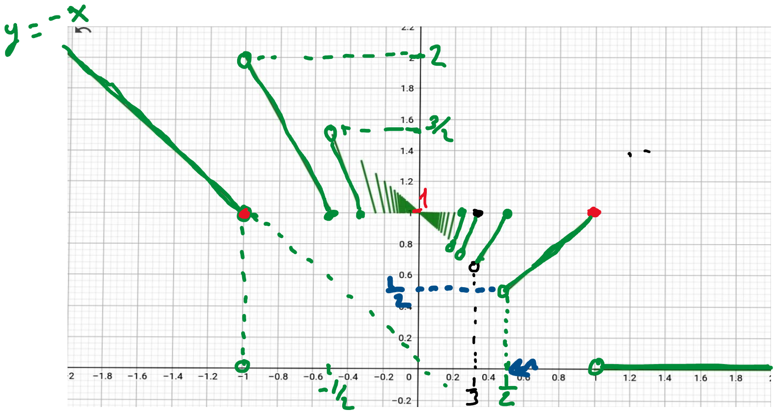
Say $h(x) = \cos x$ & $g(x) = x^2$. Since they are cont everywhere, $h \circ g (x) = f(x)$ is so.

b) $f(x) = \sqrt{1 + \sin x}$ Exc!

Q2) Let the function $f(x)$ be defined as $f(x) = \begin{cases} x \cdot [\![\frac{1}{x}]\!] & x \neq 0 \\ 1 & x = 0 \end{cases}$ Find all pts at which $f(x)$ is discontinuous.

$[\![x]\!] :=$ the greatest integer $n \leq x$ & it is discontinuous at $a \in \mathbb{Z}$

For $[\![\frac{1}{x}]\!]$, we should check the pts of the form $\mp \frac{1}{n}$, $n \in \mathbb{N}$ as well.



If $x > 1$, $x \cdot [\![\frac{1}{x}]\!] = 0$

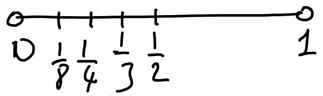
If $x < -1$, $x \cdot [\![\frac{1}{x}]\!] = x \cdot (-1) = -$

$\Rightarrow \frac{1}{x} > -1$ i.e, $y = -x$

If $x = +1$, $f(1) = 1$

If $x = -1$, $f(-1) = -1 \cdot -1 = \underline{\underline{1}}$

If $0 < x < 1 \Rightarrow f(x) = ?$



$f(\frac{1}{2}) = \frac{1}{2} \cdot [\![\frac{1}{\frac{1}{2}}]\!] = \frac{1}{2}$

$\lim_{x \to \frac{1}{n}} f(x)$ $n \in \mathbb{N}$

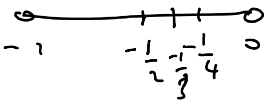
$\lim_{x \to \frac{1}{n}^+} x \cdot [\![x]\!] = \frac{1}{n} \cdot (n-1) = \underline{\underline{1 - \frac{1}{n}}}$

$x > \frac{1}{n} \Rightarrow n > \underbrace{\frac{1}{x}}$ & x is very close to $\frac{1}{n}$; $\frac{1}{x}$ is very close to n

$\lim_{x \to \frac{1}{n}^-} x \cdot [\![x]\!] = \frac{1}{n} \cdot n = 1$

$x < \frac{1}{n} \Rightarrow n < \frac{1}{x}$ & $\frac{1}{x}$ close to n & x is very close to $\frac{1}{n}$

If $-1 < x < 0 \Rightarrow f(x) = ?$



$\lim_{x \to \frac{1}{n}} f(x) =$ $\underline{\underline{n \in \mathbb{Z}^-}}$

$\lim_{x \to \frac{1}{n}^{+}} x \cdot [\![x]\!] = \frac{1}{n} \cdot (n-1) = 1 - \frac{1}{n}$

$x > \frac{1}{n}$

$n \cdot x < 1$ as $\underline{\underline{n < 0}}$

$\underline{n > \frac{1}{x}}$ as $x < 0$

$\lim_{x \to \frac{1}{n}^-} x \cdot [\![x]\!] = \frac{1}{n} \cdot n = 1$

Exc!

$S :=$ The set of discontinuous pts of $f(x) = \{ \frac{1}{n} \mid n \in \mathbb{Z} \}$

But, $f(x)$ is continuous

$\mathbb{R}-\}$. In particular, it is continuous at $x=0$

$$\frac{1}{x}-1 \le [\![\tfrac{1}{x}]\!] \le \frac{1}{x}+1 \xrightarrow[\text{by } x>0]{\text{multiply}} \underbrace{1-x}_{} \le x[\![\tfrac{1}{x}]\!] \le \underbrace{1+x}_{}$$

as $x\to0^+$ $\to 1$

By Squeeze thm, $\lim\limits_{x\to0^+} x[\![\tfrac{1}{x}]\!] = \underline{1}$

Similarly, $\lim\limits_{x\to0^-} f(x) = 1$ & $\lim\limits_{x\to1} f(x) = 1 = f(1)$ ✓

Q3) Suppose that f satisfies $\boxed{f(x+y)=f(x)+f(y)}$ $(*)$ for any x,y & f is cont at $x=0$. Prove that f is continuous <u>everywhere</u>

As f is <u>cont. at</u> $x=0 \iff \boxed{\lim\limits_{x\to0} f(x) = f(0) = 0}$

Also, $\underset{f(0)}{\overset{\shortparallel}{f(0+0)}} = f(0)+f(0) \implies \boxed{f(0)=0}$

Given any $a\in\mathbb{R}$.

Observe that $f(x-a)+f(a)=f(x)$ by $(*)$. Then,

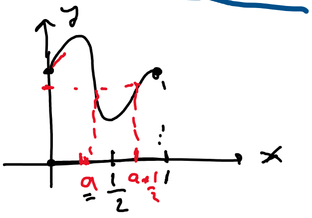
$$\lim_{x\to a} f(x) = \lim_{x\to a}\left(f(x-a)+f(a)\right) \underset{J}{=} \lim_{u\to0}\left(\cancel{f(u)}^{0}+f(a)\right) = f(a)$$

$u=x-a$

$u\to0$ as $x\to a$

$\therefore$ f is cont everywhere $\square$

Q4) Let $f:[0,1]\longrightarrow\mathbb{R}$ be cont. func. which satisfies $f(0)=f(1)$. Prove that there exists a number $a\in[0,\tfrac{1}{2}]$ s.t $\underline{f(a)=f(a+\tfrac{1}{2})}$.



Consider $g(x) = f(x+\tfrac{1}{2}) - f(x)$

$g(0) = f(\tfrac{1}{2}) - f(0)$

$g(\tfrac{1}{2}) = f(1) - f(\tfrac{1}{2}) = -\left(f(\tfrac{1}{2}) - f(1)\right)$

we know that $f(0)=f(1)$

If $f(\tfrac{1}{2})=f(0)$, then $g(0)=g(\tfrac{1}{2})=0$. Then, we <u>are done</u>.

If $f(\tfrac{1}{2})\neq f(0)$, then $g(0)\cdot g(\tfrac{1}{2})<0$

As $g(x)$ is cont. func. on $[0,\tfrac{1}{2}]$ and $g(\tfrac{1}{2})\cdot g(0)<0$, by Intermediate Value Thm, $\exists$ $a\in(0,\tfrac{1}{2})$ s.t.

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$f(a)=0$. That is, $f(a)=f(a+\frac{1}{2})$ for some $a\in(0,\frac{1}{2})$ □

Recall: I.V.T f is cont. on $[a,b]$ (w.l.o.g. $f(a)>f(b)$). If $y\in[f(b),f(a)] \Rightarrow \exists\, c\in(a,b)$ s.t. $f(c)=y$.

Q5) Given $f(x)=\begin{cases}1+x\cos\frac{2\pi}{x} & x>0\\ x^2+1 & x\le 0\end{cases}$. Show that $\exists\, c\in[-1,2$ s.t. $f(c)=0$

$[-1,4$

$f(-1)=2$
$f(4)=1$ but $f(-1)\cdot f(4$

$f(-1)=2>0$

$f(2)=1+2.(-1)=-1<0$

$$\lim_{x\to0^+}(1+\overset{0\,(*)}{x.\cos\frac{2\pi}{x}})=1=\lim_{x\to0^-}x^2+1 \Rightarrow \lim_{x\to0}f(x)=1=f(1)$$

∴ f is cont at $x=($

(*) [$-x \le x\cos(\frac{2\pi}{x}) \le x$

as $x\to0^+$ $\quad x>0$

0

By Squeeze thm,

$\lim_{x\to0^+} x\cos(\frac{2\pi}{x})=0$]

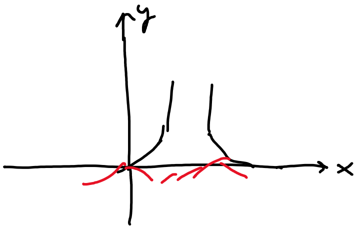
Actually, f is cont everywhere

Away from $x=0$, it is combination of continuous functions.

In particular, f is cont on $[-1,2]$ & $f(2).f(-1)<0$, by I.VT, we can say that $\exists\, c\in(-1,2)$ s.t. $f(c)=0$. □

(we can say $\exists c\in(-1,2)\subsetneq\underline{\underline{(-1,4)}}$ s.t. $f(c)=0$ although $f(4)>0$.

Q6) Show that the func. $f(x)=x^3-15x+1$ has three roots (zeros) on $[-4,4$



$f(-4)=-3<0$
$f(0)=1>0$
$f(1)=\underline{-13}<0$
$f(4)=5>0$

Apply I.V.T on each of the closed intervals $[-4,0]$, $[0,1]$ and $[1,4]$

As f is a poly. func., it is cont. everywhere. Especially, it is so on each of $[-4,0]$, $[0,1]$ & $[1,4]$.

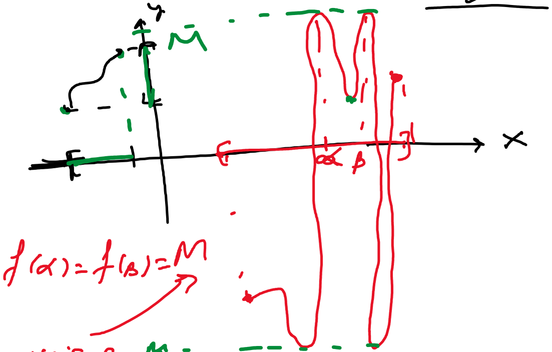
Since f is cont. on $[-4,0]$ and $f(-4).f(0)<0$, by I.V.T,
$\exists\ c_1 \in (-4,0)$ s.t $f(c_1)=0$

In a similar manner, we can find $c_2 \in (0,1)$ & $c_3 \in (1,4)$
s.t $f(c_2)=f(c_3)=0$. Exc!

$\because$ c_i's are different to each other.

By I.V.T, f has at least 3 real roots. By fund. thm of alg., f has at most 3 real roots. $\therefore$ f has exactly 3 real roots.

Q7) Show that if f is cont. func. defined on a closed inter $[a,b]$, then the range of f is also closed interval.



As f is cont. on closed & bdd interval $[a,b]$, by Extreme Value thm, it attains the abs. max. M abs. min. m.

But, it takes this value M at several pts, e.g. α, β

Then, there exists c & d in $[a,b]$ s.t. $f(c)=M$ and $f(d)$

w.l.o.g. $c<d$.

Claim: $f([a,b])=[m,M]$

$\supseteq$: $e \in [m,M]$. Then, as f is cont on $[c,d] \subset [a,b]$ and $f(c)=M \geq e$
& $f(d)=m \leq e$, by I.V.T, $\exists\ e_1 \in [c,d]$ s.t. $f(e_1)=e$.
$\therefore$ $e \in f([a,b])$.

$\subseteq$: Let $e \in f([a,b])$. Then, $\exists\ e_1 \in [a,b]$ s.t. $f(e_1)=e$.
Then, $m \leq f(e_1)=e \leq M$. $\therefore$ $e \in [m,M]$.

$\therefore$ $f([a,b])=[m,M]$.

Q8) Determine whether following statements are True OR

a) f, g funcs. on $\mathbb{R}$. If f & g are discont. at b, then $f+g$

FALSE

Consider $f(x)=\begin{cases} \frac{|x|}{x} & x \neq 0 \\ 1 & x=0 \end{cases}$, $g(x)=\begin{cases} -\frac{|x|}{x} & x \neq 0 \\ -1 & x=0 \end{cases}$ which are discont. at $x=$

Check!

But, $(f+g)(x)=0$ which is a Constant func. on $\mathbb{R}$. $\therefore$ It is cont

$\because$ [illegible]

$f(x) = \frac{1}{x}$ is cont everywhere as its domain is $D_f = \mathbb{R} - \{0\}$ ...where

But $f(x) = \begin{cases} \frac{1}{x} & x \neq 0 \\ 153 & x = 0 \end{cases}$ is not cont. at $\underline{x=0}$

b) If f cont. everywhere, then $|f(x)|$ is cont everywher

TRUE

Given any $a \in \mathbb{R}$

Assume f is cont. at $x=a \Rightarrow \lim_{x\to a} f(x) = f(a)$

Claim: $\lim_{x \to a} |f(x)| = |f(a)|$

Given $\varepsilon > 0$. Choose $\delta = \delta_0 > 0$

If $0 < |x-a| < \delta$,

$$\Big|\underbrace{|f(x)|}_{\alpha} - \underbrace{|f(a)|}_{|-f(a)| = \beta}\Big| \le |f(x) - f(a)| \underset{(*)}{<} \varepsilon$$

by triangle ineq.

$$\big||\alpha| - |\beta|\big| \le |\alpha + \beta|$$

(*) As $\lim_{x\to a} f(x) = f(a)$, for the given ε, $\exists$ a $\delta_0 > 0$

s.t. ~~$0<$~~ $|x-a| < \delta_0 \Rightarrow |f(x) - f(a)| < \varepsilon$

as it is cont

So, $|f(x)|$ is cont. at $\underline{x=a}$.

c) $\lim_{x\to c} f(x)$ & $f(c)$ exist $\Rightarrow f$ is cont. at c.

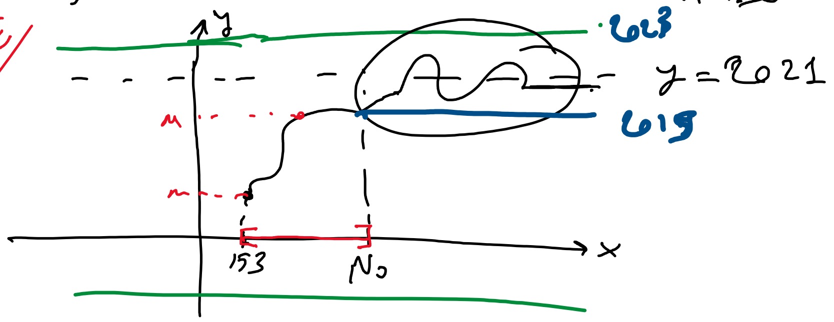
FALSE

Consider $f(x) = \begin{cases} x & x \neq 0 \\ 153 & x = 0 \end{cases}$

$\lim_{x\to 0} x = 0 \neq f(0) = 153$ although $f(0)$ exists.

d) $f: [153, \infty) \to \mathbb{R}$ cont. & $\lim_{x\to\infty} f(x) = 2021 \Rightarrow f$ is bound

TRUE



$\lim f(x) = 2021 \Rightarrow \forall \varepsilon > 0, \exists N > 0$ s.t. $x > N \Rightarrow |f(x) - 202$

$x \to \infty$

Choose $\varepsilon = 2$. $\exists\ \underline{N_0} > 0$ s.t. $x > N_0 > 153 \Rightarrow |f(x) - 2021$

$\Rightarrow \underline{2019} < f(x) < \underline{2023}$

Also, $f : \underset{\text{closed \& bdd}}{[153, N_0]} \longrightarrow \mathbb{R}$, by E.V.T, f attains

the abs. max. M & min. m.

$\therefore\ f : [153, \infty) \longrightarrow \mathbb{R}$ lies between $\underbrace{\min\{2019, m\}}_{\alpha}$ &

$\underbrace{\max\{2023, M\}}_{\beta}$. $\therefore$ It is $\underline{\underline{\text{bounded.}}}$ $\square$

(i.e. $|f(x)| \leq \max\{|\alpha|, |\beta|\}$)