Week 1
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7:47 PM

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Recitation Activities (20pts)

HWs (10pts) | Quizzes (10pts)

$\# Q$'s $= 6$

Q1 Let $x, y \in \mathbb{R}$ such that $-3 < x < 5$ and $-2 < y < 1$.

a) Find the largest and smallest possible integer values of x^2+6x+2.

$-3 < x < 5 \Rightarrow 0 \le x^2 < 25$

$\Rightarrow -18 \le 6x < 30$

$+$

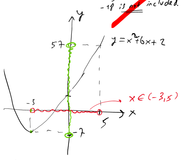
$-18 < x^2+6x+2 < 55$. So, $-16 < x^2+6x+2 < 57$

-18 is included.

an upper bound of x^2+6x+2

a lower bound of x^2+6x+2

for $x \in (-3,5)$. So, -17 (56) may not be the smallest (largest) integer value of x^2+6x+2.



From geometric viewpoint, $x^2+6x+2 \in (-7, 57)$.

Soln: $x^2+6x+2 = (x+3)^2-7$

$-3<x<5 \overset{(\text{imply})}{\Longrightarrow} 0 < x+3 < 8 \Rightarrow 0 < (x+3)^2 < 64 \Rightarrow -7 < (x+3)^2 - 7 < 57$

$\therefore$ (Therefore) The largest/smallest possible integer value is 57 / -6

b) Find the largest and smallest possible integer value of $xy-y$.

1st way: $xy - y = y(x-1)$

$$\left.\begin{array}{l} -4 < x-1 < 4 \\ -2 < y < 1 \end{array}\right\} \Rightarrow \underset{-8}{-24} < (x-1)\cdot y < -4\cdot(-2) = 8$$

$\therefore$ The largest/smallest possible integer value is 7 / -9.

2nd way: $-2<y<1 \Rightarrow -1 < -y < 2$ (i.e. $2 > -y > -1$) (Watch the order!)

Warning!

$$\left.\begin{array}{l} -3 < x < 5 \\ -2 < y < 1 \end{array}\right\} \begin{array}{l} -10 < xy < 6 \\ -1 < -y < 2 \end{array}$$

To make $xy-y$ max., xy must be very close to 6 (as y must be very close to -2) and y is very close to -1. But, for the same y, these are 2 conditions. So, we cannot say $-11 < xy - y < 8$

$xy - y = (x-1)y$

$3<x<5 \Rightarrow -4 < x-1 < 4$

$-2<y<1$. Then,

$-2 < y < 0$ OR $y = 0$ OR $0 < y < 1$

Case i: If $y \in (-2,0)$, $\underset{8}{-4y}\,(x-1)\cdot y \underset{(x)}{>} -8$

(x) [As $-2 < y < 0$, $-8 < 4y < 0$] $\Rightarrow \boxed{8 > (x-1)\cdot y > -8}$ $\rightarrow$ max $= 7$, min $= -7$

Case ii) If $y = 0$, $(x-1)\cdot y \boxed{= 0}$ for all x. $\rightarrow$ max $=$ min $= 0$

Case iii: If $y \in (0,1)$,

$\boxed{-4 < -4y < (x-1)\cdot y < 4y < 4}$ $\Rightarrow$ max $\rightarrow 3$, min $\rightarrow -3$

After comparing the values above, we can say that the largest/smallest integer value is 7 / -7

Q2 Let $x, y \in \mathbb{R}$ s.t. $x<y$. Determine which of the following statements always hold(s):

a) $x^2 < y^2$

False: $-7 < 2$ but $(-7)^2 \not< 2^2$ $\Rightarrow$ a counterexample

b) $\frac{1}{x} > \frac{1}{y}$

$-1 < 2$ but $\frac{1}{-1} \not> \frac{1}{2}$ $\Rightarrow$ a counterex.

c) $x^3 < y^3$

True: $x^3 - y^3 = (x-y)(x^2+xy+y^2) = \underbrace{(x-y)}_{<0 \text{ as } x<y \text{ our assumption}} \cdot \underbrace{(\underbrace{x^2+y^2}_{0 \text{ as at least one of } x,y \text{ is non-zero}} + \underbrace{(xy)}_{x^2+y^2>xy})}_{x^2+y^2>xy} \cdot \frac{1}{2} < 0$

$\therefore x^3 < y^3$ $\square$ $\Leftarrow$ we complete the proof.

Q3 Find the values of $x \in \mathbb{R}$ s.t. $|x+2| + |x-1| = 153$

Sign changing pts: $x=-2$ & $x=1$

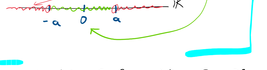
	-2	1	$\mathbb{R}$
Case i: If $x \in (-\infty,-2]$, $-(x+2)-(x-1) = 153$. $\Rightarrow -2x-2+1 = 153$ $\Rightarrow -2x = 154$ $\Rightarrow \boxed{x = -77} < -2$	If $x \in (-2,1)$, $x+2-(x-1) = 153$ $\Rightarrow 0+3 = 153$ $\Rightarrow 0 \neq 150$ No soln. in this interval.	If $x \in [1,\infty)$ $(x+2)+(x-1) = 153$. $\Rightarrow 2x+1 = 153 \Rightarrow 2x = 152$ $\Rightarrow \boxed{x = 76} > 1$	

$\therefore$ The soln. set is $\{-77, 76\}$.

Q4 Solve the following inequalities:

a) $|x+3| - 5 \ge x$

Recall: $a>0$
- $|x| > a \Rightarrow x > a$ OR $x < -a$ ($|x| = |-x|$)
- $|x| < a \Rightarrow -a < x < a$ ($|x-a| < a$)



Then, $|x+3| \ge x+5$. So, $x+3 \ge x+5$ OR $(x+3) \le -(x+5)$

Case i: $x+3 \ge x+5 \Rightarrow 0 \ge 2$ $\times$ Impossible. No soln.

Case ii: $x+3 \le -(x+5) \Rightarrow -x-3 \ge x+5 \Rightarrow -8 \ge 2x$ $\Rightarrow \boxed{-4 \ge x}$

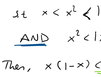
$\therefore$ Soln. set $= (-\infty, -4]$

b) $|x^2-7| < 2$

Then, $-2 < x^2 - 7 < 2 \Rightarrow 5 < x^2 < 9$. Then,

$\cancel{-3 < x < 3}$ since $-3 < -\sqrt{5}$

$x \in (-3,-\sqrt{5})$ OR $(\sqrt{5}, 3)$



So, the soln set is $(-3,-\sqrt{5}) \cup (\sqrt{5}, 3)$.

Q5 Find the interval of $x \in \mathbb{R}$ st $x < x^2 < |x|$.

$x < x^2 < |x|$ means that $x < x^2$ AND $x^2 < |x|$

If $x < x^2$, $x - x^2 < 0$. Then, $x(1-x) < 0$.

$x(1-x)$: $-$ | 0 | $+$ | 1 | $-$

Then, $x < 0$ OR $x > 1$, i.e. $\boxed{(-\infty, 0) \cup (1, \infty)}$

If $x^2 < |x|$ (i.e. $|x| > x^2$), then $x > x^2$ OR $x < -x^2$

Then, if $x > x^2$, $x(1-x) > 0$

$x(1-x)$: $-$ | 0 | $+$ | 1 | $-$ $\Rightarrow x \in (0,1)$

if $x < -x^2$, then $x(1+x) < 0$

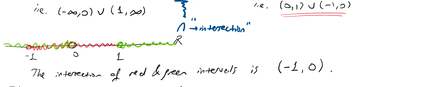
$x(1+x)$: $+$ | -1 | $-$ | 0 | $+$ $\Rightarrow x \in (-1, 0)$

After combining these data, we get

$(x < 0$ OR $x > 1)$ AND $(0 < x < 1$ OR $-1 < x < 0)$

i.e. $(-\infty, 0) \cup (1, \infty)$ $\qquad$ i.e. $(0,1) \cup (-1,0)$

$\cap$ "intersection"



The intersection of red & green intervals is $(-1, 0)$.

Thus, the soln set is $(-1, 0)$. $\rightarrow$ an "open interval."

Q6 Find the domain and range of each func. & sketch their graphs (using shifting & scaling).

a) $f(x) = \sqrt{x^2+1}$, Dom $f = \mathbb{R}$ as $x^2+1 \ge 0$ &

Range of $f = [1, \infty)$ since $x^2+1 \ge 1$ & $\sqrt{x^2+1} \ge 1$

$y = f(x) = \sqrt{x^2+1} \Rightarrow y^2 = x^2+1 \Rightarrow y^2 - x^2 = 1$

$\Rightarrow$ It is a hyperbola