Week 12

Friday, June 11, 2021 10:39 AM

Q1) Parametrize the curve of intersection of the given surfaces.

a) The elliptic cylinder $\underbrace{\frac{x^2}{9}+\frac{y^2}{4}=1}$ with the surface $\underline{z=xy}$

Let $x=3\cos\theta$ and $y=2\sin\theta$. Then, $z=3.2.\cos\theta.\sin\theta$

$$\Gamma(t)=\langle 3\cos\theta, 2\sin\theta, 3\sin 2\theta\rangle, \quad \theta\in[0,2\pi]$$

b) The cone $x^2+y^2=z^2$ with $\frac{x}{y}=\underline{\tan z}$

$$\begin{cases} x^2+y^2=z^2 \\ x=y\tan z \end{cases} \Rightarrow \begin{cases} x=y\tan z \\ y^2\tan^2 z+y^2=z^2 \end{cases} \Rightarrow \begin{cases} x=y\tan z \\ y^2=\frac{z^2}{1+\tan^2 z}=\frac{z^2}{\sec^2 z}. \\ \quad =z^2.\cos^2 z \end{cases}$$

Then, $y=\mp z.\cos z \Rightarrow x=\mp z\cos z.\tan z=\mp z.\sin z$

Say $\underline{z=t}$. Then, $\underline{y=t.\cos t \Rightarrow x=t\sin t}$

OR $y=-t\cos t \Rightarrow x=-t\sin t$

$\therefore \underbrace{\Gamma(t)}_{\text{"intersection curve"}}=\Gamma_1(t)\sqcup\Gamma_2(t)$ where

$\Gamma_1(t)=\langle t\sin t, t\cos t, t\rangle$ and $\Gamma_2(t)=\langle -t\sin t, -t\cos t, t\rangle$

for $t\in\mathbb{R}-\{(2k+1)\frac{\pi}{2} \mid k\in\mathbb{Z}\}-\{0\}$

o/w: $\tan t$ is not defined

o/w: $\frac{x}{y}$ is not defined as $y=\mp t\cos t\neq 0$

$\Updownarrow$

$t\neq 0$ & $t\neq(2k+1)\frac{\pi}{2}$ $\forall k\in\mathbb{R}$

c) The paraboloid $z=4x^2+y^2$ & the parablic cylinder $y=x^2$

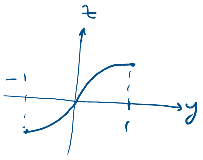
exc!

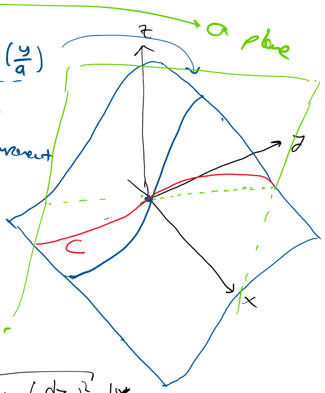
Q2) Describe the parametric curve C given by $x=a\cos t\sin t$, $y=a\sin 2t$, $z=bt$. What is the length of C between $t=0$ and $t=T>0$.

$$x=\frac{a}{2}\sin 2t, \quad y=a\sin 2t, \quad z=bt \Rightarrow \Gamma(t)=\langle\underbrace{\frac{a}{2}\sin 2t}_{x}, \underbrace{a\sin 2t}_{y}, \overset{z}{bt}\rangle \quad t\in[0,T]$$

$$\begin{cases} y=2x \\ \frac{1}{2}\arcsin\left(\frac{y}{a}\right)=\frac{z}{b} \end{cases} \Rightarrow \begin{cases} y=2x \\ \underline{z=\frac{b}{2}\arcsin\left(\frac{y}{a}\right)} \end{cases}$$

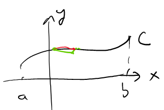
→ a surface & its x-component is free





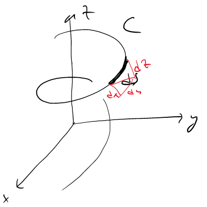
C: the curve of intersection of the surfaces!

Remark:



$$l(c)=\int_a^b ds=\int_a^b\sqrt{1+\left(\frac{dy}{dx}\right)^2}\,d\cancel{x}$$

$$\Rightarrow ds=\sqrt{dx^2+dy^2}=\sqrt{1+\left(\frac{dy}{dx}\right)^2}\,dx$$



$$\ell(C) = \underbrace{\int_C 1.\,ds}_{\text{line integral}} = \underbrace{\int_a^b \|r'(t)\|\,dt}_{\text{definite integral}}$$

$$ds = \sqrt{dx^2+dy^2+dz^2} = \sqrt{\left(\frac{dx}{dt}\right)^2+\left(\frac{dy}{dt}\right)^2+\left(\frac{dz}{dt}\right)^2}\,dt$$

where $C: r(t) = \langle x(t), y(t), z(t)\rangle$
$t \in [a,b]$

Then, $\ell(C) = \int_C ds = \int_0^T \sqrt{x'(t)^2+y'(t)^2+z'(t)^2}\,dt$

$$= \int_0^T \sqrt{a^2\cos^2 2t + 4a^2\cos^2 2t + b^2}\,dt = \int_0^T \sqrt{5a^2\cos^2 2t+b^2}\,dt = \dots$$ exc!

Q3) Parametrize the intersection of the sphere $x^2+y^2+z^2=1$ and the elliptic cylinder $x^2+2z^2=1$. Find the total length of this intersection curve.

$$\left\{\begin{array}{l} i)\ x^2+y^2+z^2=1 \\ ii)\ x^2+2z^2=1 \end{array}\right. \Rightarrow \left.\begin{array}{l} x^2 = 1-y^2-z^2 \\ x^2 = 1-2z^2 \end{array}\right\}$$ Solve the eqns at the same time!

$$\Rightarrow 1-y^2-z^2 = 1-2z^2 \Rightarrow -y^2 = -z^2 \Rightarrow y^2 = z^2 \Rightarrow \begin{array}{c} y=z \\ \text{OR} \\ y=-z \end{array} \quad (*)$$

from ii), $x^2 = 1-2z^2$. Let $x = \cos\theta$, $z = \frac{1}{\sqrt{2}}\sin\theta$, $\theta \in [0,\pi]$

Then, $y = \frac{1}{\sqrt{2}}\sin\theta$ OR $-\frac{1}{\sqrt{2}}\sin\theta$ by $(*)$

$$C: r(\theta) = \underbrace{\langle \cos\theta, \tfrac{1}{\sqrt{2}}\sin\theta, \tfrac{1}{\sqrt{2}}\sin\theta\rangle}_{C_1:\ r_1(\theta)} \cup \underbrace{\langle \cos\theta, -\tfrac{1}{\sqrt{2}}\sin\theta, \tfrac{1}{\sqrt{2}}\sin\theta\rangle}_{C_2:\ r_2(\theta)}$$

$, \theta \in [0, 2\pi]$

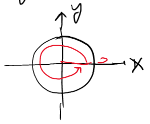
$$\ell(C) = \underbrace{\ell(C_1)+\ell(C_2)}_{\text{have equal lengths}} = 2\ell(C_1) = 2\int_{C_1} ds$$

! $r_1'(\theta) = \langle -\sin\theta, \frac{1}{\sqrt{2}}\cos\theta, \frac{1}{\sqrt{2}}\cos\theta\rangle$

$$= 2\int_0^{2\pi} \sqrt{\sin^2\theta + \tfrac{1}{2}\cos^2\theta + \tfrac{1}{2}\cos^2\theta}\,d\theta$$

$$= 2\int_0^{2\pi} 1.\,d\theta = 2.2\pi = 4\pi$$

! $x^2+y^2=1 \rightarrow$ curve in $\mathbb{R}^2$



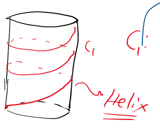
$r(\theta) = \langle \cos\theta, \sin\theta\rangle$

$\theta \in [0, 2\pi]$

~~$[0, \pi]$~~

o/w: you find $2.\ell(C_1)$.

$x^2+y^2=1 \longrightarrow$ cylinder in $\mathbb{R}^3$



C_1: $r(t) = \langle \overset{x}{\cos t}, \overset{y}{\sin t}, \overset{z}{t}\rangle$

$t \in [-\infty, \infty]$

$t = \boxed{\arcsin y = z}$

$\boxed{x^2+y^2=1}$

C_1: intersection of the surfaces

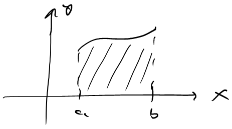
$$\int_{-\infty}^{\infty} \|r'(t)\|\,dt \longleftarrow \int_{C_1} ds$$

by applying parametrization. a line integral

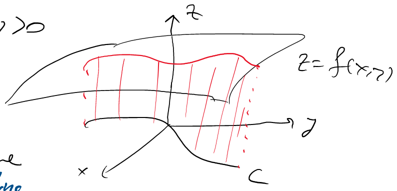
Q4) Evaluate $I = \underline{\int_C x+y \, ds}$ where C is the straight line segment $x=t$, $y=1-t$, $z=0$ from $(0,0,0)$ to $(1,0,0)$

$C: r(t) = \langle t, 1-t, 0\rangle$, $t \in [0,1] \Rightarrow r'(t) = \langle 1,-1,0\rangle$

! $\int_a^b f(x)\,dx$ for $f(x)>0$

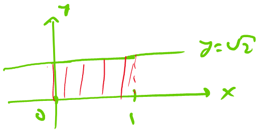


$\int_C \overset{= x+y}{f(x,y)}\, ds$ for $f(x,y)>0$

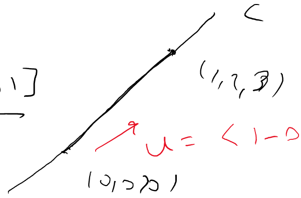


pass from the line integral to a definite integral by parametrizing the curve $\underline{C}$!

$$\int_0^1 \|r'(t)\|\,dt = \int_0^1 \sqrt{1^2+(-1)^2+0^2}\;dt = \int_0^1 \underline{\sqrt{2}\,dt} = \sqrt{2}$$



Q5) Find $\int_C x e^{yz}\, ds$ where C is the line segment from $(0,0,0)$ to $(1,2,3)$.



$u = \langle 1-0, 2-0, 3-0\rangle = \langle 1,2,3\rangle$

$C: r(t) = (0,0,0) + t\langle 1,2,3\rangle$, $\underline{t \in [0,1]}$

$= \langle t, 2t, 3t\rangle$

$$\therefore \int_C x e^{yz}\, ds = \int_0^1 t.e^{2t.3t} . \|r'(t)\|\,dt \quad \text{where } r'(t) = \langle 1,2,3\rangle$$

$$= \int_0^1 t e^{6t^2} \sqrt{1^2+2^2+3^2}\,dt = \sqrt{14}\int_0^1 t e^{6t^2}\,dt = \sqrt{14}\,\frac{e^{6t^2}}{12}\Big|_0^1 = \underline{\frac{\sqrt{14}}{12}(e^6-1)}$$

Q6) Evaluate $I = \int_C \frac{x^2}{y^{4/3}}\, ds$ where C is the curve $x=t^2$, $y=t^3$ for $t\in[1,2]$

$C: r(t) = \langle t^2, t^3\rangle \Rightarrow r'(t) = \langle 2t, 3t^2\rangle$. Then,
$t \in [1,2]$

$$I = \int_1^2 \frac{t^4}{t^{\frac{4.3}{3}}}\, \|r'(t)\|\,dt = \int_1^2 \frac{t^4}{t^4}\sqrt{(2t)^2+(2t^2)^2}\,dt = \int_1^2 \sqrt{4t^2+9t^4}\,dt$$

$$= \int_1^2 t\sqrt{4+9t^2}\,dt = \ldots\ldots$$ exc!

Q7) Find $I = \int_C \sqrt{1+4x^2z^2}\, ds$ where C is the curve of intersection of surfaces $x^2+z^2=1$ & $y=x^2$

$C: \begin{cases} x^2+z^2=1 \\ y=x^2 \end{cases}$ Let $x=\cos\theta$ & $z=\sin\theta$. Then, $y=\cos^2\theta$ $\theta\in[0,2\pi]$

$C: r(\theta) = \langle \cos\theta, \cos^2\theta, \sin\theta \rangle$, $\theta\in[0,2\pi]$

$$I = \int_0^{2\pi} \sqrt{1+4\cos^2\theta\,\sin^2\theta}\; \|r'(\theta)\|\, d\theta$$ where $r'(\theta) = \langle -\sin\theta, -2\cos\theta\sin\theta, \cos\theta \rangle$

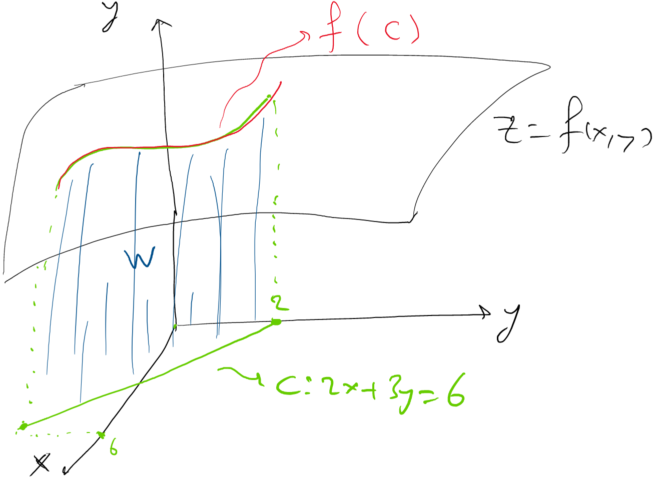
$$= \int_0^{2\pi} \sqrt{1+(\sin 2\theta)^2}\, \sqrt{\underline{\sin^2\theta} + \sin^2 2\theta + \underline{\cos^2\theta}}\, d\theta$$

$$= \int_0^{2\pi} \sqrt{1+\sin^2(2\theta)}\, \sqrt{1+\sin^2 2\theta}\, d\theta = \int_0^{2\pi} 1+\sin^2 2\theta\, d\theta$$

$$= \int_0^{2\pi} 1\, d\theta + \int_0^{2\pi} \underbrace{\sin^2 2\theta}_{\frac{1-\cos 4\theta}{2}}\, d\theta = 2\pi + \left(\frac{\theta}{2} - \frac{\sin 4\theta}{2.4}\right)\Bigg|_0^{2\pi} = 3\pi$$

exc!

Q8) Find the area of one side of "wall" standing orthogonally on the curve $C: 2x+3y=6$, $0\le x\le 6$ and beneath the curve on the surface $f(x,y) = 4+3x+2y$



$$A(W) = \int_C f(x,y)\, ds$$ "line integral"

(! $f(x,y) > 0$ for $(x,y)\in C$)

$C: \begin{cases} z = 4+3x+2y \\ 2x+3y=6 \end{cases} = \begin{cases} y = \frac{6-2x}{3} \\ z = 4+3x+2\left(\frac{6-2x}{3}\right) \end{cases}$ $x\in[0,6]$

Then, $C: r(t) = \langle t, \frac{6-2t}{3}, 8+\frac{5}{3}t \rangle$, $t\in[0,6]$

Then, $A(W) = \int_0^6 \left(4+3.t+2.\left(\frac{6-2t}{3}\right)\right).\|r'(t)\|\, dt$ where

$r'(t) = \langle 1, -\frac{2}{3}, \frac{5}{3} \rangle$. Then,

$$A(W) = \int_0^6 \left(8+\frac{5}{3}t\right).\sqrt{\frac{38}{9}}\, dt = \sqrt{\frac{38}{9}} \int_0^6 8+\frac{5}{3}t\, dt$$

$$= \frac{\sqrt{38}}{3}\left(8t+\frac{5t^2}{6}\right)\Bigg|_0^6 = 26\sqrt{38}$$