Week 11

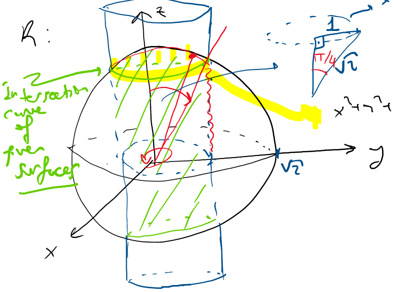
Friday, June 4, 2021 10:36 AM

From the last week;

Q4) Let R be the region in $\mathbb{R}^3$ bdd by $x^2+y^2 \le 1$ & $x^2+y^2+z^2 \le 2$

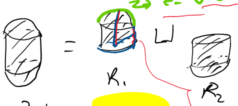
Express the volume of R using an iterated (triple) integral in spherical coordinates.

R :



$x^2+y^2=1$ & $z=1$

R :



$\Rightarrow z=\sqrt{2-x^2-y^2}$ its radius is $\sqrt{2}$

$x^2+y^2+z^2=2$ & $x^2+y^2=1 \Rightarrow z=\mp 1$

$x^2+y^2=1$

$\Downarrow$

$\rho^2 \sin^2\phi(\cos^2\theta+\sin^2\theta)=1$

$\Downarrow$

$\rho=\frac{1}{\sin\phi}$ as $\rho>0$

$V(R)=2.\ V(R_1)$

$$=2.\iiint_{R_1} 1.\,dV$$

$$=2.\left[\int_0^{2\pi}\int_0^{\pi/4}\int_0^{\sqrt{2}} \rho^2\sin\phi\, d\rho\, d\phi\, d\theta+\int_0^{2\pi}\int_{\pi/4}^{\pi/2}\int_0^{1/\sin\phi}\rho^2\sin\phi\, d\rho\, d\phi\, d\theta\right]$$

v( )

v( )

This week we will solve some questions in Chapter 12, 13 & 14:

Review:

Q1) Let $f(x,y)=\begin{cases}\frac{x^3}{x^2+y^2} & \text{if } (x,y)\neq(0,0)\\ 0 & \text{if } (x,y)=(0,0)\end{cases}$

Calculate each of $f_1(0,0)$ ✓, $f_2(0,0)$, $f_{12}(0,0)$ ✓ & $f_{21}(0,0)$

$$f_1(0,0)=\lim_{h\to 0}\frac{f(0+h,0)-f(0,0)}{h}=\lim_{h\to 0}\frac{\frac{h^3}{h^2}-0}{h}=1$$

$$f_1(x,y)=\frac{3x^2.(x^2+y^2)-x^3.2x}{(x^2+y^2)^2} \quad \text{if } (x,y)\neq(0,0)$$

Then, $$f_{12}(0,0)=\lim_{h\to 0}\frac{f_1(0,0+h)-f_1(0,0)}{h}=\lim_{h\to 0}\frac{\frac{0}{h^4}-1}{h}$$

$$=\lim_{h\to 0}\frac{1}{h}=\text{"d.n.e"}$$

$f_2(0,0)$ & $f_{21}(0,0)$ are exc!

Q2) Let $f(x,y,z)=(x^2+z^2)\sin\frac{\pi xy}{2}+yz^2$. Let $P_0=(1,1,-1)$.

a) Find the gradient of f at P_0.

$$\nabla f\Big|_{P_0}=\left\langle 2x.\sin\left(\frac{\pi xy}{2}\right)+(x^2+z^2).\frac{\pi y}{2}.\cos\frac{\pi xy}{2},\ \frac{\pi x}{2}(x^2+z^2).\cos\frac{\pi xy}{2}+z^2,\ 2z\sin\frac{\pi xy}{2}+2yz\right\rangle\Big|_{P_0}=\langle 2,1,-4\rangle$$

b) Find the linearization of f at P_0.

$f(x)=x^2$ ⇝ a curve in $\mathbb{R}^2$ "tangent line"

$f(x)=x^2$ ⇝ a surface in $\mathbb{R}^3$ "tangent plane"

$f(x,y)$? in $\mathbb{R}^4$ " ?

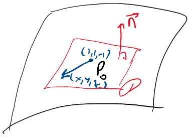
$h(x,y,z)=x^2$ ⇝ ......

$$L_{P_0}(f(x,y,z)) = f(1,1,-1) + f_1(1,1,-1)\cdot(x-1) + f_2(1,1,-1)(y-1) + f_3(1,1,-1)\cdot(z+1)$$

$f(x,y,z)$

Here, we know that $\nabla f(P_0) = \langle f_1, f_2, f_3\rangle \big|_{P_0} = \underline{\langle 2, 1, -4\rangle}$

c) Find an eqn. for the tangent plane at P_0 to the level surface of f through P_0.



$f(x,y,z)=k \Rightarrow f(P_0)=k \Rightarrow \boxed{k=3}$

$f(1,1,-1) = 2+1 = 3$

So, this is the level surface $f(x,y,z)=3$.

$\vec{n} \parallel \nabla f(P_0) \Rightarrow$ choose $\vec{n} = \nabla f(P_0) = \langle 2,1,-4\rangle$

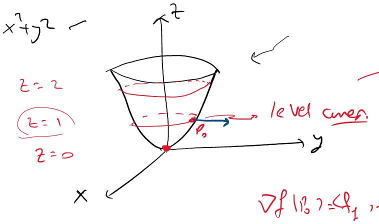
Then, $\mathcal{P}: \langle x-1, y-1, z+1\rangle \cdot \langle 2,1,-4\rangle = 0 \Rightarrow 2(x-1)+y-1-4z+4=0$

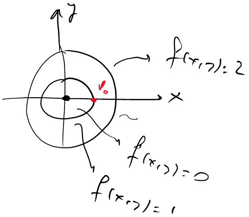
$\therefore \mathcal{P}: 2x+y-4z=7$

Remark:

$z = x^2+y^2$

$f(x,y)$

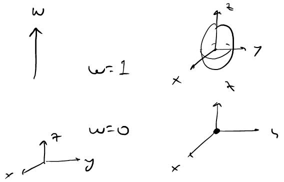




$\nabla f(P_0) = \langle f_1, f_2\rangle \big|_{P_0}$

$w = x^2+y^2+z^2 : \mathbb{R}^3 \longrightarrow \mathbb{R}$

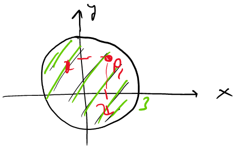
$f(x,y,z)$



$\nabla f(P_0) = \langle f_1, f_2, f_3\rangle \big|_{P_0}$

Q3 Find the extreme values of $f(x,y) = x^2+y^2-3x-xy$ on the disk $x^2+y^2 \leq 9$

D:



(As f is Cont. on a closed & bdd
| it attains abs. max. & min.

Inside D:

$\nabla f(P) = \langle 0,0 \rangle$ if P is a critical pt of f.

$f_x = 2x-3-y=0 \Rightarrow 4y-3-y=0 \Rightarrow \boxed{y=1}$

$f_y = 2y-x=0 \Rightarrow \boxed{x=2y}$

$\Downarrow$ $\boxed{x=2}$

$P_1 = (1,2) \in D$ ∴ It is a critical pt. of f
& $\nabla f(P_1)=0$

$f(P_1) = f(2,1) = 5+1-6-2=-2$ i.e. $\boxed{f(P_1)=-2}$

" No singular pt " as f is diff'ble

for the classification of P_1, $f_{xx}=2$, $f_{yy}=2$, $f_{xy}=-1$

Then, $H_f(P) = \begin{vmatrix} 2 & -1 \\ -1 & 2 \end{vmatrix} = 4-1 = 3 > 0$

$f_{xx}f_{yy}-f_{xy}^2$

$-\Delta > 0 \Rightarrow \boxed{\Delta < 0}$ → $f_{xx}=2>0$

P_1 is loc. min

Boundary D: $x^2+y^2=9$

Method of Lagrange multipliers

$L = f + \lambda . g$ where $f(x,y) = x^2+y^2-3x-xy$ subject to $x^2+y^2=9$

$g(x,y) = x^2-y^2-9=0$

$f_x = \lambda g_x$
$f_y = \lambda g_y$
$g(x,y)=0$

$\Rightarrow$

$2x-3-y = \lambda . 2x \Rightarrow 2x(1-\lambda) = 3+y$ (i)

$2y-x = \lambda . 2y \Rightarrow 2y(1-\lambda) = x$ (ii)

$x^2+y^2=9$ (iii)

Then, $1-\lambda = \boxed{\frac{3+y}{2x} = \frac{x}{2y}}$ (*) if $x \neq 0$, $y \neq 0$

o/w: $x=0 \Rightarrow y=-3 \Rightarrow \lambda = 1 \Rightarrow \boxed{(0,-3)=P_2}$
by i

$y=0 \Rightarrow x=0$ but $x^2+y^2 \neq 9$ ↯
by ii

$(*) \Rightarrow 6y+2y^2 = 2x^2 \Rightarrow x^2=y^2+3y$. By (iii), $9-y^2=y^2+3y$

$\Rightarrow 2y^2+3y-9=0 \Rightarrow y=-3$ OR $\boxed{y=+3/2}$

$\Downarrow$ by (iii)

$$x = 0$$
$$\Downarrow$$
$$\lambda = 1 \qquad \underline{(0,-3) = P_2}$$

$y = \frac{3}{2} \Rightarrow x = \frac{3\sqrt{3}}{2} \Rightarrow 1-\lambda = \sqrt{3}/2 \Rightarrow$ the eqn (i), $2x(1-\lambda) \overset{?}{=} 3+y$ ✓

$\Downarrow$ OR by (iii) $\quad \searrow$ by ii)

exc!

$x = -\frac{3\sqrt{3}}{3} \Rightarrow 1-\lambda = \sqrt{3}/2 \Rightarrow$ Check the eqn. $2x(1-\lambda) \overset{?}{=} 3+y$ ✓

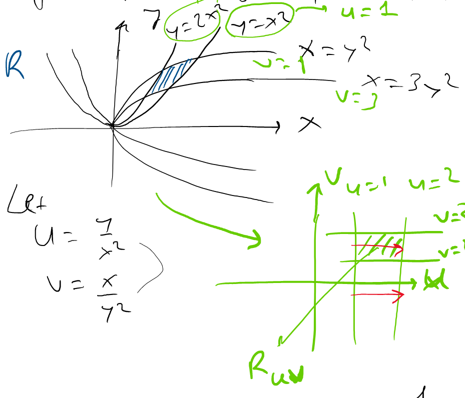
$$P_3 = \left(\frac{3\sqrt{3}}{2}, \frac{3}{2}\right) \ \& \ P_4 = \left(-\frac{3\sqrt{3}}{2}, \frac{3}{2}\right)$$

Observe that $f(P_2) = 9$, $f(P_3) = 9 - \frac{27\sqrt{3}}{4}$, $f(P_4) = 9 + \frac{27}{4}\sqrt{3}$

$$f(P_1) = -2$$

$$-2.7 \approx \underbrace{f(P_3)}_{\text{abs min of } f \text{ over } D} < \underbrace{f(P_1)}_{\sim -2} < f(P_2) < \underbrace{f(P_4)}_{\text{abs max of } f \text{ over } D}$$

Q4) Find the area of the finite plane region bdd by the four parabolas $y = x^2$, $y = 2x^2$, $x = y^2$ & $x = 3y^2$.



$$A(R) = \iint_R 1 \, dA$$

Let
$u = \frac{y}{x^2}$
$v = \frac{x}{y^2}$

$$= \int_1^3 \int_1^2 1 \cdot \left|\frac{\partial(x,y)}{\partial(u,v)}\right| \underline{du\,dv}$$

change of variable

where
$$\frac{\partial(x,y)}{\partial(u,v)} = \frac{1}{\frac{\partial(u,v)}{\partial(x,y)}} = \frac{1}{\begin{vmatrix} u_x & u_y \\ v_x & v_y \end{vmatrix}} = \frac{1}{\begin{vmatrix} -\frac{2y}{x^3} & \frac{1}{x^2} \\ \frac{1}{y^2} & -\frac{2x}{y^3} \end{vmatrix}} = \frac{1}{\frac{4xy}{x^3y^3} - \frac{1}{x^2y^2}}$$

$$= \frac{x^3y^3}{3xy} = \frac{x^2y^2}{3} = \frac{1}{3u^2 \cdot v^2} \Rightarrow \left|\frac{1}{3u^2v^2}\right| = \frac{1}{3u^2v^2} \text{ as } \underline{u^2 \cdot v^2 > 0}$$

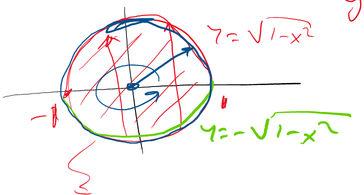
as $x \cdot y \neq 0$ $\quad$ 3 $\quad$ 2

Then, $A(R) = \iint_{1\ 1} \frac{1}{3u^2v^2}\, du\, dv$

$$= \int_1^3 \frac{1}{3v^2}\cdot \frac{u^{-1}}{-1}\Big|_1^2 dv = \frac{1}{6}\int_1^3 \frac{1}{v^2}\, dv = \dots \quad \boxed{= \frac{1}{9}}$$

Q5 $\displaystyle\int_{-1}^{1}\int_{y=-\sqrt{1-x^2}}^{y=\sqrt{1-x^2}} \frac{2\,dy\,dx}{(1+x^2+y^2)^2} = \iint_R \frac{2\,dA}{(1+x^2+y^2)^2}$ where

R:



$y=\sqrt{1-x^2} \Rightarrow \boxed{y^2+x^2=1}$

a disc: $x^2+y^2 \le 1$

$$= \int_0^{2\pi}\int_0^1 \frac{\boxed{2\cdot r\,dr}\,d\theta}{(1+r^2)^2}$$

$$= \int_0^{2\pi}\int_1^2 \frac{du\,d\theta}{u^2} = \int_0^{2\pi} \frac{u^{-1}}{-1}\Big|_1^2 d\theta$$

$u = 1+r^2$

$du = 2r\,dr$

$= \dots \quad \boxed{= \pi}$

exc!

Remark! $\displaystyle\int \cos^2\theta\, d\theta = \int \frac{\cos 2\theta + 1}{2}\, d\theta$

by using $\cos^2\theta - \sin^2\theta = \cos 2\theta$

$\displaystyle\int \sin^2\theta\, d\theta = \int \frac{1-\cos 2\theta}{2}\, d\theta$

Q6 $f(x,y) = \begin{cases} \dfrac{xy^3}{x^2+y^6} & (x,y)\neq(0,0) \\ 0 & (x,y)=(0,0) \end{cases}$ ✓

a) Find $D_v f(0,0)$ for any unit vector $v=\langle a,b\rangle$

$$D_v f(0,0) = \lim_{h\to 0} \frac{f(0+a.h,\, 0+b.h) - f(0,0)}{h} = \lim_{h\to 0} \frac{\dfrac{a.bh^4}{h^2(a^2+b^6h^4)} - 0}{h}$$

as $v=\langle a,b\rangle$

$$= \lim_{h\to 0} \frac{ab\cancelto{0}{h}}{a^2+\cancelto{0}{b^6h^4}} = 0 \quad \text{if } a\neq 0$$

If $a = 0$, $D_v f(0,0) = \lim\limits_{h \to 0} \dfrac{0 \quad = 0}{h} = 0$

Then, $D_v f(0,0) = 0$ for any unit vector $v = \langle a, b \rangle$.

b) Is it diff'ble?

$$\nabla f(0,0) = \left\langle \lim_{h \to 0} \frac{f(0+h,0) - f(0,0)}{h}, \lim_{h \to 0} \frac{f(0,0+h) - f(0,0)}{h} \right\rangle$$

$$\underset{\substack{| \\ \text{exc!}}}{=} \langle 0, 0 \rangle$$

$\therefore$ $\nabla f(0,0) \cdot v = 0$ for any $v = \langle a, b \rangle$ & $D_v f(0,0) = \langle 0, 0 \rangle$.

But, this does not say whether it is diff'ble or not!

What is the remedy?

No! It is not even cont. at $(0,0)$

$$\left.\begin{aligned} \lim_{(x,y) \to (0,0)} f(x,y) &\underset{\substack{| \\ \text{along } \Gamma_1 : \, x = 0}}{=} \lim_{y \to 0} \frac{0}{y^6} = 0 \\ \lim_{(x,y) \to (0,0)} f(x,y) &\underset{\substack{\backslash \\ \text{along } \Gamma_2 : \, x = y^3}}{=} \lim_{y \to 0} \frac{y^6}{2y^6} = \frac{1}{2} \end{aligned}\right\} \neq$$

$\therefore$ This limit d.n.e

$\therefore$ It is not cont at $(0,0)$

So, it is not diff'ble.