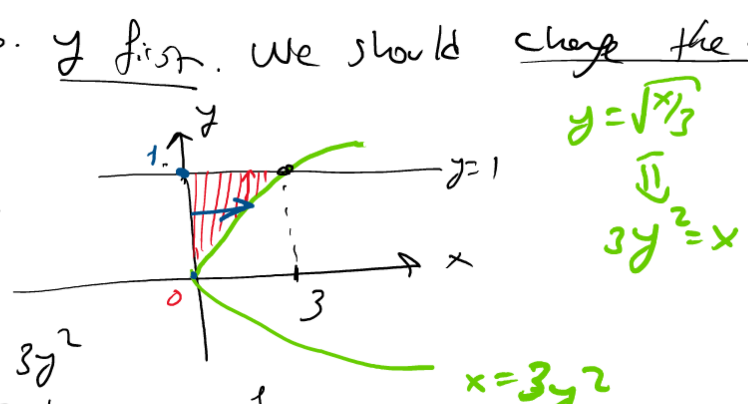
Week 10

Friday, May 28, 2021 10:39 AM

Q1) Evaluate the iterated integral $I=\int_0^3\int_{\sqrt{x/3}}^1 e^{y^3}\,dy\,dx$.

- It is hard to integrate w.r.t. y first. We should change the order

$I=\iint_R e^{y^3}\,dA$ where R:



$$=\int_0^1\int_0^{3y^2} e^{y^3}\,dx\,dy=\int_0^1 x.e^{y^3}\Big|_0^{3y^2}dy=\int_0^1 3y^2.e^{y^3}dy$$

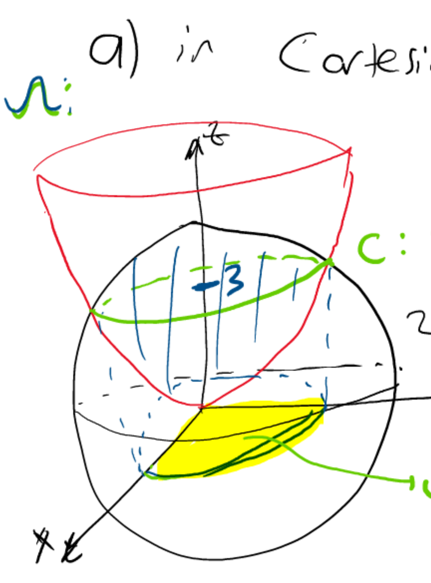
$x=0 \quad x=3y^2$

$$=e^{y^3}\Big|_0^1=\underline{e-1}$$

Q2) Consider the space region Ω in the first octant (i.e. $x\geq0, y\geq0, z\geq0$) which lies inside the sphere $x^2+y^2+z^2=12$ and above the Paraboloid $z=x^2+y^2$. Express the triple integral $I=\iiint_\Omega f(x,y,z)\,dV$ as an iterated triple integral.

a) in Cartesian coordinates (x,y,z)

Ω:



C: curve of intersection: $12=x^2+y^2+z^2$ & $z=x^2+y^2$

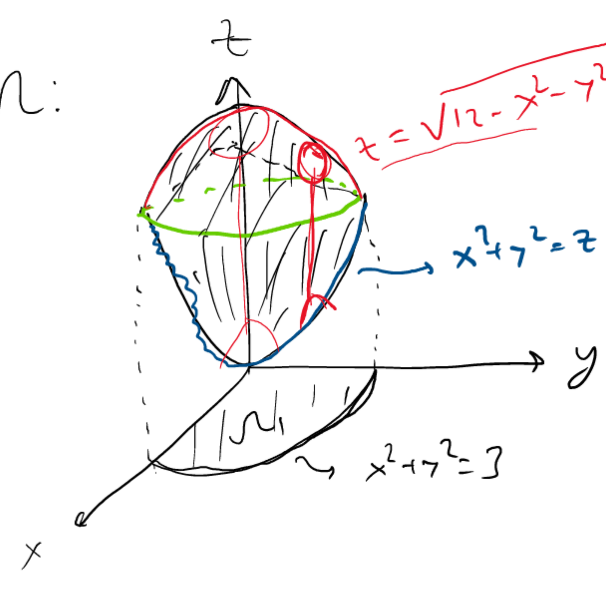
$\rightarrow x^2+y^2+z^2=12$

$z^2+z-12=0 \Rightarrow \boxed{z=3}$ OR ~~$z=-4$~~

Then, C: $x^2+y^2=3,\ z=3$ i.e. a circle lying in the plane $z=3$

Ω_1: projection of Ω on the xy-plane

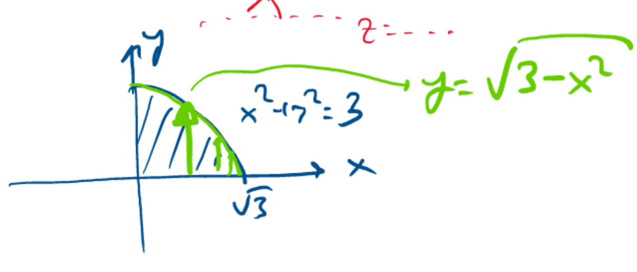
Ω:



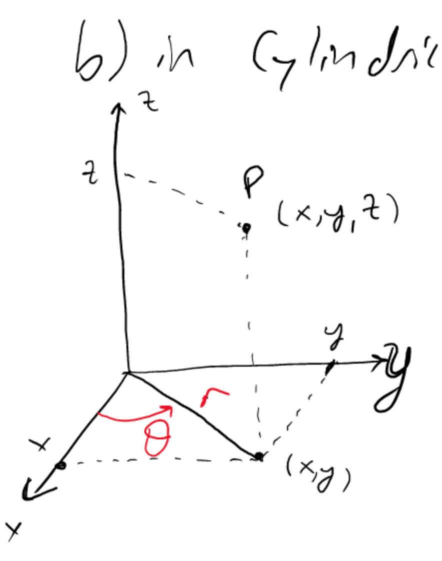
$$I=\int_0^{\sqrt3}\int_0^{\sqrt{3-x^2}}\int_{x^2+y^2}^{\sqrt{12-x^2-y^2}} f(x,y,z)\,dz\,dy\,dx$$

focus on Ω_1

Ω_1:



b) in Cylindrical coordinates (r,θ,z)



$x=r\cos\theta$
$y=r\sin\theta$, $r>0$
$z=z$
$dV=r\,dz\,dr\,d\theta$
$x^2+y^2=r^2$

$$I=\int_0^{\pi/2}\int_0^{\sqrt3}\int_{r^2}^{\sqrt{12-r^2}} f(r\cos\theta, r\sin\theta, z).r\,dz\,dr\,d\theta$$

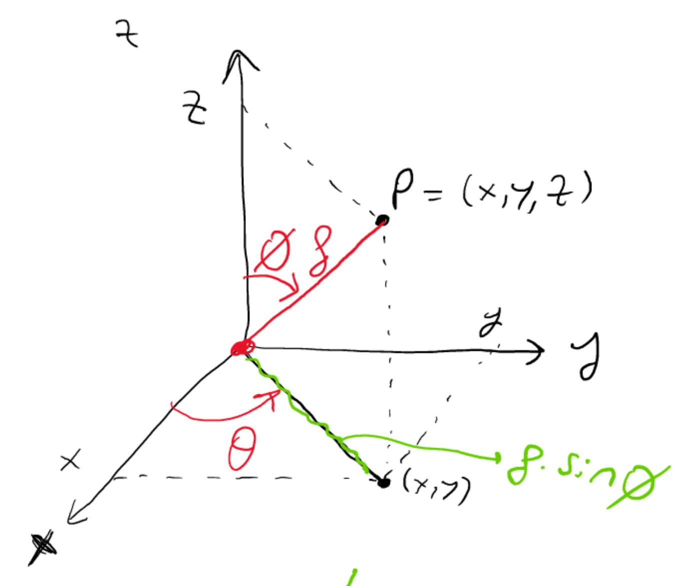
Ω_1

$z=\sqrt{12-r^2}$

$z=r^2$

Ω_1:

c) in Spherical coordinates (ρ,ϕ,θ)

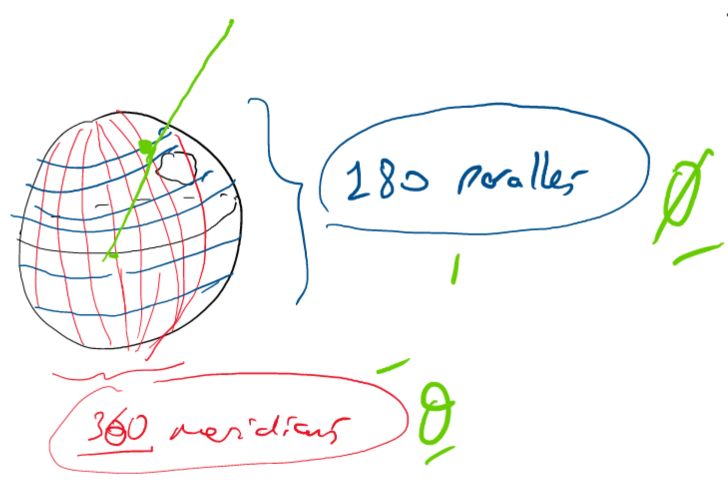


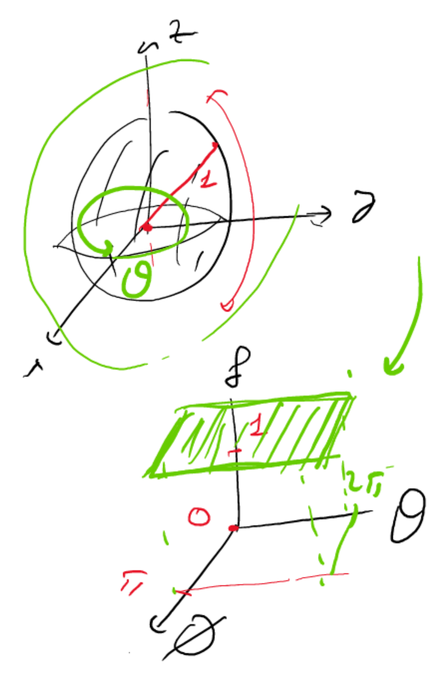
$x=\rho.\sin\phi.\cos\theta$
$y=\rho.\sin\phi.\sin\theta$
$z=\rho.\cos\phi$

$dV=\rho^2\sin\phi\,d\rho\,d\phi\,d\theta$

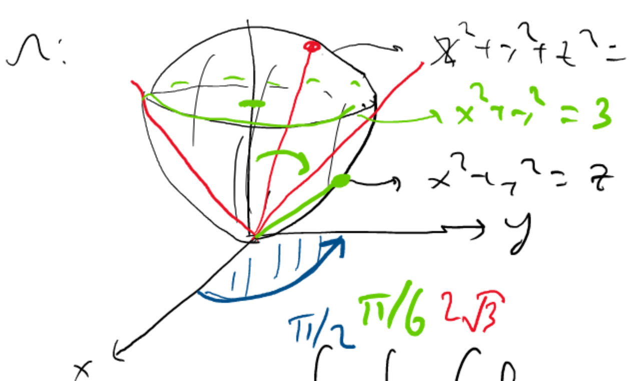
$\rho>0$, $\phi\in[0,\pi]$, $\theta\in[0,2\pi]$

$x^2+y^2+z^2=\rho^2$

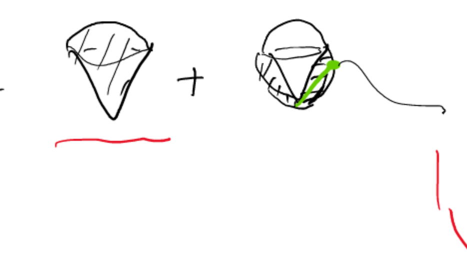


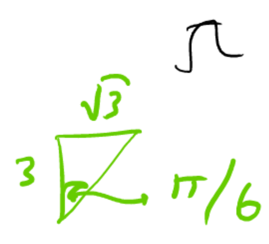


Ω:



$\Omega =$





$z=x^2+y^2$
⇕
$\rho\cos\phi=\rho^2\sin\phi(\sin^2\theta+\cos^2\theta)$
⇕
$\rho(\cos\phi-\rho\sin^2\phi)=0$
⇕
~~$\rho=0$~~ OR $\rho=\frac{\cos\phi}{\sin^2\phi}$

$$I=\int_0^{\pi/2}\int_0^{\pi/6}\int_0^{2\sqrt3} f(\rho\sin\phi\cos\theta,\rho\sin\phi\sin\theta,\rho\cos\phi).\rho^2\sin\phi\,d\rho\,d\phi\,d\theta+$$

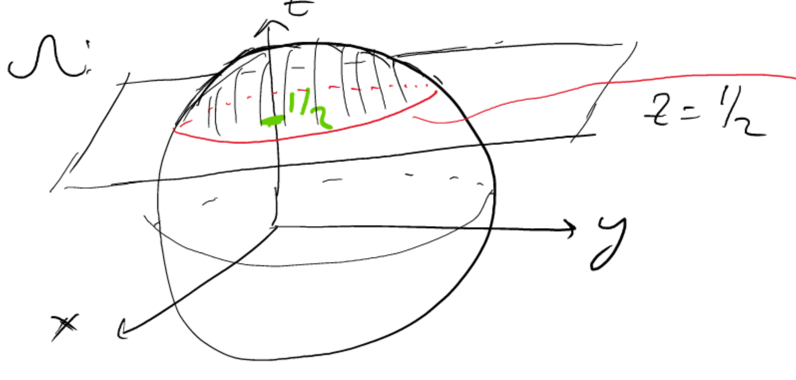
$$\int_0^{\pi/2}\int_{\pi/6}^{\pi/2}\int_0^{\frac{\cos\phi}{\sin^2\phi}} f(\ -\ -\ -\ -\ )\rho^2\sin\phi\,d\rho\,d\phi\,d\theta$$

Q3) Consider the space region $\Omega=\{(x,y,z)\in\mathbb{R}^3: z\geq\frac12,\ x^2+y^2+z^2\leq1\}$

In other words, Ω is the region lying inside the unit sphere $x^2+y^2+z^2=1$ and above the plane $z=\frac12$. Express the volume of region Ω

a) as an iterated double integral in Cartesian coordinates (x,y).

Ω:

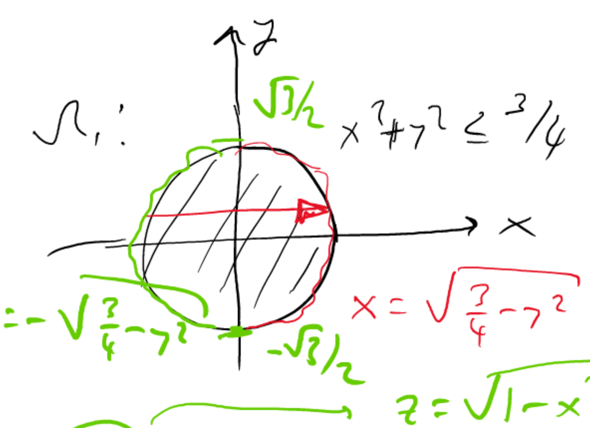


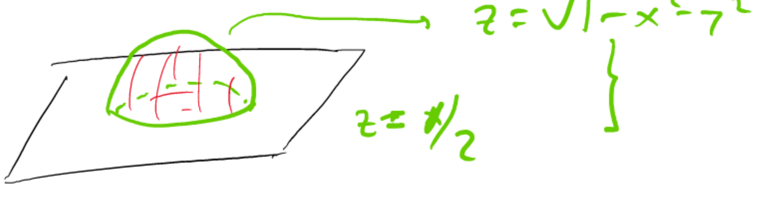
C: $z=\frac12$ and $x^2+y^2+z^2=1$

$\Rightarrow x^2+y^2=\frac34$ circle on the plane $z=\frac12$

Ω_1: projection of Ω on xy-plane

Ω_1:





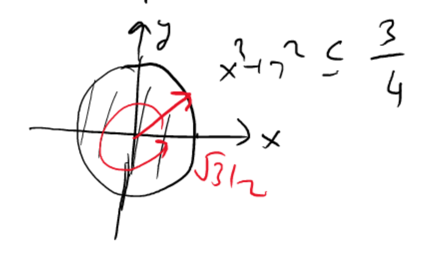
$$V(\Omega)=\iint_{\Omega_1}\sqrt{1-x^2-y^2}-\frac12\,dA$$

$$=\int_{-\sqrt3/2}^{\sqrt3/2}\int_{-\sqrt{\frac34-y^2}}^{\sqrt{\frac34-y^2}}\sqrt{1-x^2-y^2}-\frac12\ dx\,dy$$

b) as an iterated double integral in Polar coordinates (r,θ).

$$V(\Omega)=\iint_{\Omega_1}\sqrt{1-x^2-y^2}-\frac12\,dA=\int_0^{2\pi}\int_0^{\sqrt3/2}\sqrt{1-r^2}-\frac12\ r.dr\,d\theta$$

Ω_1:



c) as an iterated triple integral in Cartesian coordinates: (x,y,z)

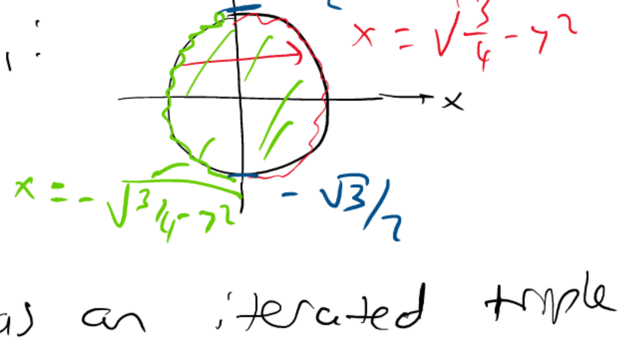
$$V(\Omega)=\iiint_\Omega 1\,dV=\int_{-\sqrt3/2}^{\sqrt3/2}\int_{-\sqrt{\frac34-y^2}}^{\sqrt{\frac34-y^2}}\int_{\frac12}^{\sqrt{1-x^2-y^2}} 1\ dz\,dx\,dy$$

over Ω_1

$z=\sqrt{1-x^2-y^2}$

$z=1/2$

Ω_1:



d) as an iterated triple integral in Cylindrical Coordinates: (r,θ,z)

$x=r\cos\theta$
$y=r\sin\theta$
$z=z$
$dv=r\,dz\,dr\,d\theta$

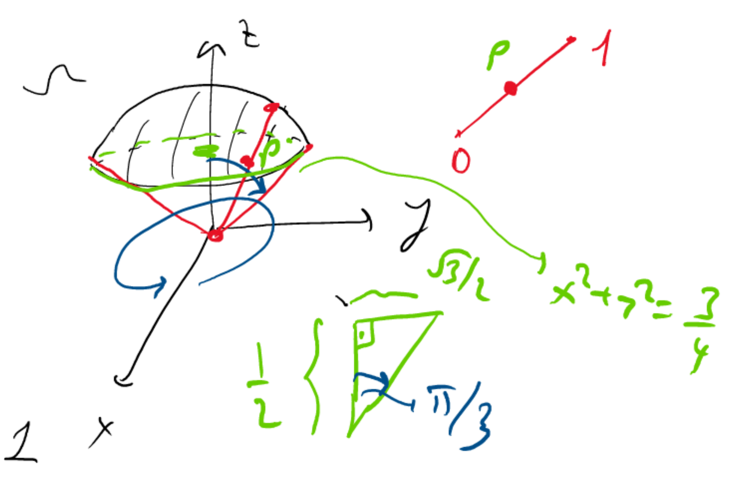
$$V(\Omega)=\iiint_\Omega 1\,dv=\int_0^{2\pi}\int_0^{\sqrt3/2}\int_1^{\sqrt{1-r^2}} 1.r\,dz\,dr\,d\theta$$

over Ω_1

e) as an iterated triple integral in Spherical Coordinates (ρ,ϕ,θ).

$x=\rho\sin\phi\cos\theta$
$y=\rho\sin\phi.\sin\theta$
$z=\rho\cos\phi$
$dv=\rho^2\sin\phi\,d\rho\,d\phi\,d\theta$

Ω



$\rho=?$

$z=\frac12=\rho\cos\phi$

$\Rightarrow \rho=\frac{1}{2\cos\phi}$

$$V(\Omega)=\int_0^{2\pi}\int_0^{\pi/3}\int_{\frac{1}{2\cos\phi}}^1 1.\rho^2\sin\phi\,d\rho\,d\phi\,d\theta$$