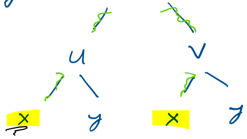
Week 7

Friday, April 30, 2021 10:38 AM

Q1) Find the indicated derivatives assuming that the func. $f(x,y)$ has cont. partial derivatives. (Then, f is diff'ble) Here, x & y are indep variables:

a) $\frac{\partial}{\partial x} f(\underbrace{y^4}_{u(x,y)}, \underbrace{x^3}_{v(x,y)})$

Say $z=f(u,v)$

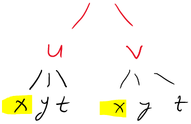


Using the chain rule,

$$\frac{\partial}{\partial x} f(y^4,x^3) = \frac{\partial f}{\partial u}\cdot \underbrace{\frac{\partial u}{\partial x}}_{=0} + \frac{\partial f}{\partial v}\cdot\underbrace{\frac{\partial v}{\partial x}}_{=3x^2} = f_2 \cdot 3x^2$$

b) $\frac{\partial}{\partial x} f(\underbrace{x^2 f(x,t)}_{u}, \underbrace{f(y,t)}_{v})$

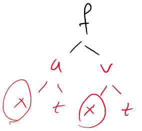
$z=f(u,v)$



by using Chain Rule

$$\overset{/}{=} f_1\cdot\frac{\partial u}{\partial x} + f_2\cdot \underbrace{\frac{\partial v}{\partial x}}_{=0} = f_1\cdot\left[2x\cdot f(x,t) + x^2\cdot\left(f_1(x,t)\cdot\underbrace{\frac{\partial x}{\partial x}}_{=1} + f_2\cdot\underbrace{\frac{\partial t}{\partial x}}_{=0}\right)\right]$$

$$\frac{\partial f(\overset{u}{t},\overset{v}{x})}{\partial x} = \frac{\partial f}{\partial u}\cdot\underbrace{\frac{\partial u}{\partial x}}_{=0} + \frac{\partial f}{\partial v}\cdot\underbrace{\frac{\partial v}{\partial x}}_{=1}$$



$$= f_2\cdot 1$$

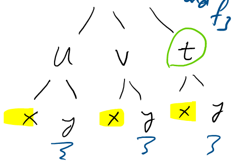
Q2) Find $\frac{\partial^2}{\partial y\partial x} f(\overset{u}{y^2}, \overset{v}{xy}, \overset{t}{-x^2})$ in terms of partial derivatives of f

$$f_{xy} = \frac{\partial}{\partial y}\left(\frac{\partial f}{\partial x}\right) = \frac{\partial^2 f}{\partial y\partial x}$$

Watch the order!!

Then, $\frac{\partial}{\partial y}\left(\frac{\partial f}{\partial x}\right) = \frac{\partial}{\partial y}\left(f_1\cdot\underbrace{\frac{\partial u}{\partial x}}_{0} + f_2\cdot\underbrace{\frac{\partial v}{\partial x}}_{y} + f_3\cdot\underbrace{\frac{\partial t}{\partial x}}_{-2x}\right)$

$z=f(u,v,t)$ & f_1, f_2 and f_3



$$= \frac{\partial}{\partial y}\left(f_2\cdot \overset{\downarrow}{y} - 2x\cdot f_3\right)$$

$$= 1\cdot f_2 + y\cdot\left[f_{21}\cdot\underbrace{\frac{\partial u}{\partial y}}_{2y} + f_{22}\cdot\underbrace{\frac{\partial v}{\partial y}}_{x} + f_{23}\cdot\underbrace{\cancel{\frac{\partial t}{\partial y}}}_{0}\right] - 2x\cdot\left[f_{31}\cdot\underbrace{\frac{\partial u}{\partial y}}_{2y} + f_{32}\cdot\frac{\partial v}{\partial y}^{=x} + f_{33}\cdot\underbrace{\cancel{\frac{\partial t}{\partial y}}}_{0}\right]$$

$$= f_2 + 2y^2 f_{21} + xy f_{22} - 4xy f_{31} - 2x^2 f_{32}$$

⚠ $f_{32} = \dfrac{\partial^2 f(u,v,t)}{\partial v\,\partial t}$

Q3) Let $f(x,y) = \begin{cases} \dfrac{x^3y - xy^3}{x^2+y^2} & \text{if } (x,y)\neq(0,0) \\ 0 & \text{if } (x,y)=(0,0) \end{cases}$

a) Find $f_x(0,0)$ & $f_y(0,0)$.

$$f_x(0,0) = \lim_{h\to 0}\frac{f(0+h,0) - f(0,0)}{h}$$

we are interested in the change in the 1st component

$$= \lim_{h\to 0}\frac{\frac{0}{h^2} - 0}{h} = 0$$

$$f_y(0,0) = \lim_{h\to 0}\frac{f(0,0+h) - f(0,0)}{h} = \lim_{h\to 0}\frac{0/h^2 - 0}{h} = 0$$

b) Show that $f_{xy}(0,0) = -1$ & $f_{yx}(0,0) = 1$

f is diff'ble away from (0,0) (exc! why?)

$$f_x(x,y) = \begin{cases} \dfrac{(3x^2y - y^3)(x^2+y^2) - (x^3y - xy^3)(2x)}{(x^2+y^2)^2} & \text{if } (x,y)\neq(0,0) \\ 0 & \text{if } (x,y)=(0,0) \end{cases}$$

$$f_y(x,y) = \begin{cases} \dfrac{-x(y^4 + 4x^2y^2 - x^4)}{(x^2+y^2)^2} & \text{if } (x,y)\neq(0,0) \\ 0 & \text{if } (x,y)=(0,0) \end{cases}$$

Then, $f_{xy}(0,0) = \lim\limits_{h\to 0}\dfrac{f_x(0,0+h) - f_x(0,0)}{h} = \lim\limits_{h\to 0}\dfrac{-\frac{h^5}{h^4} - 0}{h} = -1$

$$f_{yx}(0,0) = \lim_{h\to 0}\frac{f_y(0+h,0) - f_y(0,0)}{h} = \lim_{h\to 0}\frac{+h^5/h^4 - 0}{h} = 1$$

c) What does $f_{xy}(0,0) \neq f_{yx}(0,0)$ tell us?

Thm: If f has continuous 2nd order mixed partials at P, then
(ie. f_{xy}, f_{yx})

$$\boxed{f_{xy}(P) = f_{yx}(P)}$$

$\therefore$ By the thm, either f_{xy} or f_{yx} is not continuous at $(0,0)$.

$\lim_{(x,y)\to(0,0)} f_{xy}(x,y) \neq f_{xy}(0,0)$. In fact, $\lim_{(x,y)\to(0,0)} f_{xy}(x,y)$ d.n.e

To see this, check the following limits whose results are different!

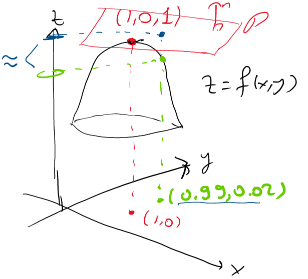
$\lim_{(x,y)\to(0,0)} f_{xy}(x,y)$ on $\Gamma_1 : x=0$

and

$\lim_{(x,y)\to(0,0)} f_{xy}(x,y)$ on $\Gamma_2 : y=0$

exc!

Q4) Use linearization to approximate the number $\sqrt{0.99}\, e^{0.02}$.



Let $f(x,y) = \sqrt{x}\cdot e^{y}$

Use the linearization of f at $(1,0)$

Observe that $f_x = \frac{1}{2\sqrt{x}} e^{y}$, $f_y = \sqrt{x}\, e^{y}$. Then,

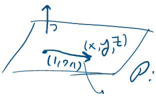
$f_x(1,0) = \frac{1}{2}$, $f_y(1,0) = 1$ & $f(1,0) = 1$

So, $f(x,y) \approx \underbrace{L(x,y) = \frac{1}{2}\cdot(x-1) + 1\cdot(y-0) + \underset{f(1,0)}{\underset{\shortparallel}{1}}}$

So, $f(0.99, 0.02) \approx \frac{1}{2}\cdot(-0.01) + 0.02 + 1 = 1.015$

Take

$\vec{n} = \langle f_x(1,0), f_y(1,0), -1 \rangle$



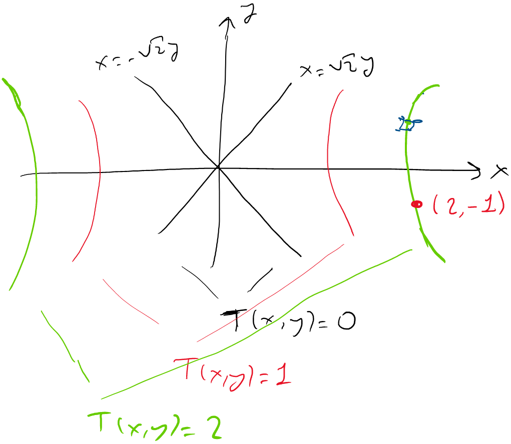
P: $(x-1, y-0, z-1)\cdot\langle f_x(1,0), f_y(1,0), -1\rangle = 0$

$f_x(1,0)(x-1) + f_y(1,0)(y-0) \underbrace{-(z-1)}_{+1-z} = 0$

$\Rightarrow \underset{L(x,y)}{\underset{\shortparallel}{z}} = f_x(1,0)(x-1) + f_y(1,0)\cdot y \boxed{+1}$

Q5) The temperature $T(x,y)$ at pts of the xy-plane is given by $T(x,y) = x^2 - 2y^2$

a) Draw a contour diagram for T showing some isotherms (curves of constant temperature)



$T(x,y)=k$ for some constant $k\in\mathbb{R}$

$k=0 \Rightarrow x^2=2y^2$ ~

$k=1 \Rightarrow x^2-2y^2=1$

$k=2 \Rightarrow x^2-2y^2=2$

} hyperbola

b) In what direction should an ant at position (2,-1) move if it wishes to cool off as quickly as possible?

Let u be the direction s.t. $\underline{D_uT(2,-1)}$ is min.

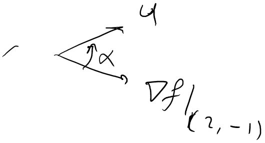
w.l.o.g. u is unit.

As T is diff'ble (as T_x & T_y are cont everywhere by Fundamental Lemma)

$$D_uT(2,-1) \underset{\substack{\Downarrow \\ T \text{ is diff'ble}}}{=} \nabla T\Big|_{(2,-1)} \cdot u = \langle T_x, T_y\rangle\Big|_{(2,-1)} \cdot u$$

$$= \langle 2x, -4y\rangle\Big|_{(2,-1)} \cdot u$$

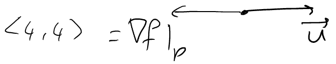
$$= \langle 4,4\rangle \cdot u = \|\langle 4,4\rangle\| \cdot \|u\| \cdot \cos\alpha$$



$= 4\sqrt{2}\cdot\cos\alpha$ is min.

$\Downarrow$

$\cos\alpha = -1 \implies \boxed{\alpha = \pi}$



Then,

$$\vec{u} = \frac{\langle -4,-4\rangle}{4\sqrt{2}} = \langle -\tfrac{1}{\sqrt{2}}, -\tfrac{1}{\sqrt{2}}\rangle$$

Q6] Consider the func. $f(x,y,z) = \frac{x}{y} + z$ at $P_0(3,1,1)$

a) Compute $\nabla f(P_0)$

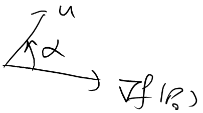
$$\nabla f(P_0) = \langle f_1, f_2, f_3\rangle\Big|_{P_0} = \langle \tfrac{1}{y}, \tfrac{-x}{y^2}, 1\rangle\Big|_{(3,1,1)} = \boxed{\langle 1,-3,1\rangle}$$

b) & c) Is there a unit vector u s.t. $D_u f(P_0) = 5$ OR 3. If yes, find one; if no, prove that it does not exist.

Since f is diff'ble at P_0 (as its partial derivatives are cont at P_0)

For any unit vector u,

$$D_u f(P_0) = u \cdot \nabla f(P_0) = \|\nabla f(P_0)\| \cdot \cos\alpha$$ where



Observe that $\|\nabla f(P_0)\| = \sqrt{1^2 + (-3)^2 + (1)^2} = \sqrt{11}$

$\therefore \quad -\sqrt{11} \le D_u f(P_0) \le \sqrt{11} < 5$ as $|\cos\alpha| < 1$

So, $\exists$ no such vector $\vec{u}$ s.t. $D_{\vec{u}} f(P_0) = 5$

As $3 \in [-\sqrt{11}, \sqrt{11}]$, we can find unit vector $\vec{u}$ s.t. $D_u f(P_0) = 3$. Say $u = \langle a,b,c\rangle$

$3 = \langle a,b,c\rangle \cdot \nabla f(P_0) = \langle a,b,c\rangle \cdot \langle 1,-3,1\rangle \iff a - 3b + c = 3$ with $a^2+b^2+c^2=1$

Take, for ex, $u = \langle 0, -1, 0 \rangle$

d) Let S be a set of all pts $P(x,y,z)$ where f increases fastest in the direction of the vector $u = \langle 2,1,2 \rangle$. Describe S geometrically

$$S \underset{\text{defn}}{:=} \left\{ P=(x,y,z) \;\middle|\; u=\langle 2,1,2\rangle \text{ is the direction making } D_u f(x,y,z) \text{ max.} \right\}$$

$\Big\Updownarrow$ as f is diff'ble

$D_u f(P) = u \cdot \nabla f(P)$ is max

i.e $\longrightarrow u$

$\longrightarrow \nabla f(P)$

i.e. $\alpha = 0$

$\Updownarrow$

$\boxed{\cos\alpha = 1}$

Then, $\nabla f(P) = \underline{\langle \frac{1}{y}, -\frac{x}{y^2}, 1 \rangle} = t \cdot \langle 2,1,2 \rangle = \underline{\langle 2t, t, 2t \rangle}$

$t > 0$

∇ ∂ $\xleftarrow{u}$ $\xrightarrow{}\nabla f$ $\Rightarrow$ $u = t \cdot \nabla f$ where $\underline{t < 0}$

Then, i) $\underline{\frac{1}{y} = 2t}$

ii) $-\frac{x}{y^2} = t$

iii) $+1 = 2t$ $\Longrightarrow$ $\boxed{t = 1/2}$ $\underset{\text{by i)}}{\Longrightarrow}$ $\boxed{y = 1}$ $\underset{\text{by ii)}}{\Longrightarrow}$ $\boxed{x = -1/2}$

Solve it together!

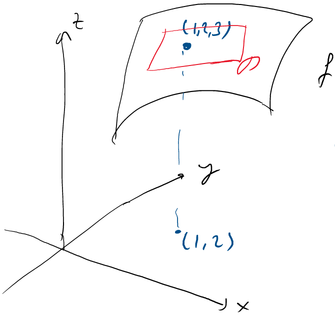
z is free i.e. $\underline{t \in \mathbb{R}}$

So, $S = \{ (-\frac{1}{2}, 1, z) \mid z \in \mathbb{R} \}$

$S: \langle x,y,z \rangle = (-\frac{1}{2}, 1, 0) + z \cdot \underline{\langle 0,0,1 \rangle}$, $z \in \mathbb{R}$ ← a parameter

$\therefore$ It is a line in $\underline{\mathbb{R}^3}$

Q7) Find the eqn of the tangent plane to the level surface of the func. $f(x,y,z) = \frac{xy}{z^2}$ at the pt $\boxed{P=(1,2,3).}$



$f(x,y,z)=\frac{2}{9}$
$\quad\;\, \| $
$\quad\;\frac{xy}{z^2}$

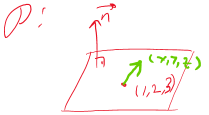
The eqn. of surface is given implicitly

$f:\mathbb{R}^3 \longrightarrow \mathbb{R}$

$\boxed{f(x,y,z)=k}$ is a level surface of $w=f(x,y,z)$ for each $\underline{k\in\mathbb{R}}$

$f(P)=f(1,2,3)=\frac{2}{9}$

So, $f(x,y,z)=\frac{2}{9}$ is the level surface of f through $P=(1,2,3)$.



$\nabla f(P) = \langle \frac{y}{z^2}, \frac{x}{z^2}, -\frac{2xy}{z^3}\rangle\Big|_P = \langle \frac{2}{9},\frac{1}{9},-\frac{4}{27}\rangle$ is perpendicular (⊥) to the level surface of f through P. i.e. $\underline{f(x,y)=\frac{2}{9}}$

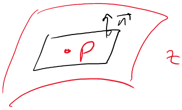
Take $\vec n = \langle 6,3,-4\rangle$

$\langle x-1, y-2, z-3\rangle\cdot\langle 6,3,-4\rangle = 0 \implies 6(x-1)+3(y-2)-4(z-3)=0$

i.e. $\boxed{6x+3y-4z=0}$

Given the surface $z=f(x,y)$.

$\implies \underline{F(x,y,z)=f(x,y)-z} \;:\mathbb{R}^3\longrightarrow\mathbb{R}$



$z=f(x,y)$ → it is given explicitly

Then $F(x,y,z)=0$ is level surface

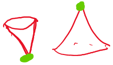
Then, $\nabla F\big|_P = \langle F_1,F_2,F_3\rangle\big|_P = \langle f_1, f_2, -1\rangle\big|_P$ ⊥ to the level surface $F(x,y,z)=0$

$\therefore\ \vec n \parallel \nabla f|_P = \underline{\langle f_1,f_2,-1\rangle}|_P$

Summary: $D_u f(P) = u\cdot\nabla f(P)$ if f is diff'ble. If f is not diff'ble, use the formal def'n of directional derivative

$$D_u f(\overset{(x_0,y_0)}{\overset{\shortparallel}{P}}) = \lim_{h\to 0^+} \frac{f(x_0+a.h,\ y_0+b.h) - f(x_0,y_0)}{h}$$

$u = \langle a,b \rangle$ $\boxed{\text{unit}}$



$$\begin{tikzcd}
 & & \text{f is cont at P_0} \arrow[dl, Rightarrow, shift left, "/" marking] \\
\text{f_x \& f_y are cont. at P_0} \arrow[r, Rightarrow, "\text{by using fund. theorem}"'] & \text{f is diff'ble at P_0} \arrow[ur, Rightarrow] \arrow[r, Rightarrow, shift left] \arrow[d, Rightarrow] \arrow[dr, Rightarrow] & \text{$f_x(P_0)$, $f_y(P_0)$ exist} \arrow[l, Rightarrow, shift left, "/" marking] \\
 & \text{Chain Rule is valid} & \text{linear approx. makes sense}
\end{tikzcd}$$

$$f = \begin{cases} 1 & \text{if } x.y = 0 \\ 2 & \text{if } x.y \neq 0 \end{cases}$$

Also, f_{xy} & f_{yx} cont at $P \Longrightarrow f_{xy}(P) = f_{yx}(P)$