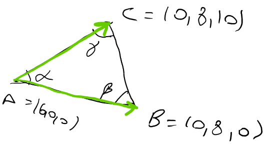
# Week 5

Friday, April 16, 2021 10:27 AM

From the last week:

Q9| Find the three angles of the triangle with vertices (6,0,0), (0,8,0) and (0,8,10).



α = ?

$\vec{AC} = \langle 0-6, 8-0, 10-0\rangle = \langle -6, 8, 10\rangle$

$\vec{AB} = \langle -6, 8, 0\rangle$

$\|\vec{AC}\| = \sqrt{6^2+8^2+10^2} = 10\sqrt{2}$ & $\|\vec{AB}\| = \sqrt{6^2+8^2+0^2} = 10$

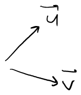
$\vec{AC}\cdot\vec{AB} = \|\vec{AC}\|\cdot\|\vec{AB}\|\cdot\cos\alpha$

$\Rightarrow \underbrace{\langle -6,8,10\rangle\cdot\langle -6,8,0\rangle} = 10\sqrt{2}\cdot 10\cdot\cos\alpha \Rightarrow \underbrace{-6\cdot -6+8\cdot 8+10\cdot 0}_{100} = 100\sqrt{2}\cos\alpha$

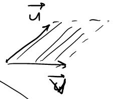
$\Rightarrow \cos\alpha = \frac{1}{\sqrt{2}} \Rightarrow \boxed{\alpha = \pi/4}$

In a similar manner, you can calculate γ & β. exc! $\left(\begin{matrix}\gamma = \pi/4\\ \beta = \pi/2\end{matrix}\right)$

Recall:



- $\vec{u} \mp \vec{v}$
- $\lambda\vec{u}$ for any $\lambda \in \mathbb{R}$
- $\vec{u} \times \vec{v} = \vec{w}$ which is perpendicular both $\vec{u}$ & $\vec{v}$ (also whose magnitude $\|\vec{u}\times\vec{v}\|$ is equal to the area of
- $\vec{u}\cdot\vec{v} = k \in \mathbb{R}$



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Q10| Let $f(x) = \sum_{n=1}^{\infty} \frac{(-1)^n \cdot n}{3^n \cdot (n+1)!} x^{2n-1}$ & $g(x)$ be the func. defined by $g(x) = f(x^3)$. Write down the power series of $g(x)$ &

. Find $g^{(45)}(0)$.

$$g(x) = f(x^3) = \sum_{n=1}^{\infty} \frac{(-1)^n \cdot n}{3^n \cdot (n+1)!} (x^3)^{2n-1} = \sum_{n=1}^{\infty} \frac{(-1)^n \cdot n}{3^n \cdot (n+1)!} x^{6n-3}$$

$\ominus 1 + \ominus x \ominus x^2 + \cdots$ $\quad n=1 \Rightarrow \frac{-1}{3.2} x^3$

$$\underbrace{\sum_{n=0}^{\infty} \frac{f^{(n)}(0)}{n!} x^n}_{\text{Taylor Series of } f(x) \text{ around } x=0} = \sum_{n=1}^{\infty} \frac{(-1)^n \cdot n}{3^n \cdot (n+1)!} x^{6n-3} = f(x)$$

as the power series of a func. f(x) at x=0 is unique! (and its power series representation about x=0 exists.)

Then,

$$\underbrace{\frac{f^{(45)}(0)}{45!}}_{\text{the 'coeff. of } x^{45}} x^{45} = \underbrace{\frac{(-1)^8 \cdot 8}{3^8 \cdot 9!}}_{\text{the coeff of } x^{45}} x^{45}$$

i.e. $6n-3=45$

$\Rightarrow \boxed{n=8}$

$$\Rightarrow f^{(45)}(0) = \frac{45! \cdot 8}{3^8 \cdot 9!}$$

What about $f^{(5)}(0)$ ?

$$\frac{f^{(5)}(0)}{5!} = \underbrace{0}_{\text{the coef of } x^5}$$

i.e. $6n-3=5$

$\Rightarrow n = 8/6 = \boxed{4/3} \notin \mathbb{N}$

$\Rightarrow \underline{f^{(5)}(0)=0}$

From this week problems!

Q1) write down vector eqn., param. eqns & symmetric eqns of the line L which contains the pt. A=(1,2,4) & is parallel to the intersection line of the two planes given by the eqns

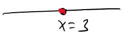
$P_1 : \textcircled{1}x + \textcircled{3}y + \textcircled{5}z = 7$ & $P_2 : -x + 2y + 3z = 4$

$1+6+20$ $\quad 1 \quad 2 \quad 4$ $\qquad 1+4+12 \neq 4$

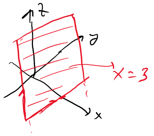
Recall:

- $x=3$ is a pt. in $\mathbb{R}$
- $x=3$ is a line in $\mathbb{R}^2$
- $x=3$ is a plane in $\mathbb{R}^3$

(x,y,z)



a line L



In $\mathbb{R}^2$

A Line L: 2 distinct pts

OR

1 pt with a slope

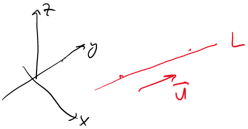
# In $\mathbb{R}^3$

For a line $L$:

2 distinct pts

OR

1 pt with a direction vector $u$.

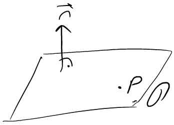


For a plane $\mathcal{P}$:

3 distinct pts

OR

1 pt. with a normal vector. $\vec{n}$



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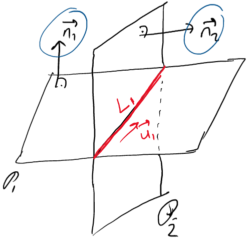
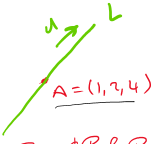




$A \notin \mathcal{P}_1$ & $\mathcal{P}_2$

If $L \parallel L_1$, then $u \parallel u_1$

$\vec{n}_1 = \langle 1, 3, 5 \rangle$

$\vec{n}_2 = \langle 2, 2, 3 \rangle$

$u_1$: direction vector of $L_1$. Then, $\vec{u}_1 \perp \vec{n}_1$ & $\vec{u}_1 \perp \vec{n}_2$. So,

$\vec{u}_1 \parallel \vec{n}_1 \times \vec{n}_2$. As $\vec{u} \parallel \vec{u}_1$, $\boxed{\vec{u} \parallel \vec{n}_1 \times \vec{n}_2}$

$$\vec{n}_1 \times \vec{n}_2 = \begin{vmatrix} i & j & k \\ 1 & 3 & 5 \\ 1 & 2 & 3 \end{vmatrix} = \langle 9-10, -(3-5), 2-3 \rangle = \langle -1, 2, -1 \rangle$$

As $\vec{u} \parallel \vec{n}_1 \times \vec{n}_2$, $\vec{u} = \lambda . \vec{n}_1 \times \vec{n}_2$ for some $\lambda \in \mathbb{R}$.

choose $\lambda = 1$.

Then, $\vec{u} = \langle -1, 2, -1 \rangle$.

$L$: $(x, y, z) = (1, 2, 4) + t \langle -1, 2, -1 \rangle$ for all $t \in \mathbb{R}$

" The vector eqn "

$$\left.\begin{aligned} x &= 1 - t \\ y &= 2 + 2t \\ z &= 4 - t \end{aligned}\right\} \quad , t \in \mathbb{R}$$

" the param. eqns "

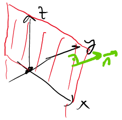
$$(t =) \ \frac{x-1}{-1} = \frac{y-2}{2} = \frac{z-4}{-1}$$

" Symm. eqn ".

Q: What is a normal vector of the plane $y = 0$? —

$0 . x + 1 . y + 0 . z = 0 \Rightarrow \vec{n} = \langle 0, 1, 0 \rangle$ is its normal vec.

~~of~~



$= (1,2,3)+t\langle 3,2,1\rangle$

Q2] For the two lines $L_1: (x,y,z) = (1+3t,\ 2+2t,\ 3+t)$, $t \in \mathbb{R}$

and $L_2: (x,y,z) = (4+t,\ 4-t,\ 5+2t)$, $t \in \mathbb{R}$

a) Show that $L_1$ & $L_2$ are skew lines:

$\vec{u}_1 = \langle 3,2,1\rangle$ a direction vector of $L_1$

$\vec{u}_2 = \langle 1,-1,2\rangle$ - - - - - - of $L_2$

→ $L_1$ is not parallel to $L_2$.

$L_1 \parallel L_2 \Rightarrow \vec{u}_1 \parallel \vec{u}_2$ i.e. $u_1 = \lambda . u_2$ for one $\lambda \in \mathbb{R}$

Assume $u_1 = \lambda . u_2$. Then, $\langle 3,2,1\rangle = \langle \lambda, -\lambda, 2\lambda\rangle$

$$\left.\begin{aligned} 3 &= \lambda \\ 2 &= -\lambda \\ 1 &= 2\lambda \end{aligned}\right\}$$ No common soln. Then, $u_1 \nparallel u_2$ ∴ $L_1 \nparallel L_2$.

"Not parallel"

→ $L_1 \cap L_2 = \emptyset$

$L_1: (x,y,z) = (1+3t,\ 2+2t,\ 3+t)$, $t \in \mathbb{R}$

$L_2: (x,y,z) = (4+s,\ 4-s,\ 5+2s)$, $s \in \mathbb{R}$

$$\left.\begin{aligned} 1+3t &= 4+s \\ 2+2t &= 4-s \end{aligned}\right\} \Rightarrow 3+5t = 8 \Rightarrow 5t = 5 \Rightarrow \boxed{t=1} \text{ Then, } s=0$$

$3+t = 5+2s$

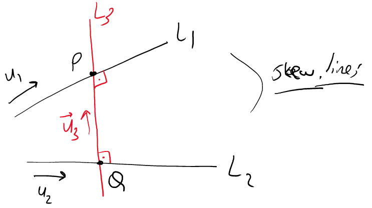
3 eqn; with 2 unknowns

After pluging $t=1$ & $s=0$ into the last equality, we get $3+1=5$

Not possible ∴ $L_1 \cap L_2 = \emptyset$ "not intersecting lines"

As two distinct lines in $\mathbb{R}^3$ are neither parallel nor intersecting they are skew!

b) Find the line $L_3$ which intersects both of the lines $L_1$ & $L_2$ perpendicularly (at seperate pts of course)



Then, $u_3 \perp u_1$ and $u_3 \perp u_2$

So, $\vec{u}_3 \parallel \vec{u}_1 \times \vec{u}_2$

$$\vec{u}_1 \times \vec{u}_2 = \begin{vmatrix} i & j & k \\ 3 & 2 & 1 \\ 1 & -1 & 2 \end{vmatrix} = \langle 5, \underbrace{-(3.2-1.1)}_{}^{-5}, -5 \rangle \text{ So, } \vec{u}_3 \parallel \langle 5,-5,-5 \rangle$$

$$\rightarrow 5\langle 1, \boxed{-1}, -1 \rangle$$

Take $\vec{u}_3 = \langle 1, -1, -1 \rangle$ for simplicity! We need to find a pt on $L_3$.

Say $P \in L_1 \cap L_3$ & $Q \in L_2 \cap L_3$. Then, they are of the form

$P = (1+3t_0, 2+2t_0, 3+t_0)$ for some $t_0 \in \mathbb{R}$ & $Q = (4+s_0, 4-s_0, 5+2s_0)$ for some $s_0 \in \mathbb{R}$.

$L_1: (x,y,z) = (1+3t, 2+2t, 3+t)$, $t \in \mathbb{R}$ $\qquad L_2:$

Then, $\overrightarrow{QP} = \langle -3+3t_0-s_0, -2+2t_0+s_0, -2+t_0-2s_0 \rangle \parallel \langle 1,-1,-1 \rangle$

Then, $\overrightarrow{QP} = \lambda \langle 1,-1,-1 \rangle$. Then,

$$\left\{ \begin{array}{l} -3+3t_0-s_0 = \lambda \\ -2+2t_0+s_0 = -\lambda \\ -2+t_0-2s_0 = -\lambda \end{array} \right. \quad -5+5t_0 = 0 \Rightarrow \boxed{t_0 = 1} \Rightarrow \boxed{\lambda = -s_0}$$

From the last eqn., $-2+1-2s_0 = s_0 \Rightarrow \boxed{s_0 = -1/3}$

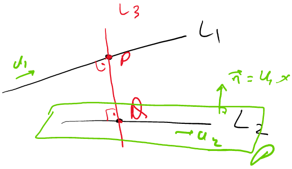
$\therefore P = (4,4,4)$ and $Q = \left(\frac{11}{3}, \frac{13}{3}, \frac{13}{3}\right)$

$\therefore L_3 : (x,y,z) = (4,4,4) + t\langle 1,-1,-1 \rangle$ for all $t \in \mathbb{R}$.

c) Find the two pts $P \in L_1$ & $Q \in L_2$ which are closest to each other. What is the distance between $L_1$ & $L_2$ in terms of the pts $P$ & $Q$?

In part b), we have found the two pts $P \in L_1$ & $Q \in L_2$ which are closest to each other.

$$\left.\begin{array}{l} P=(4,4,4) \\ Q=\left(\frac{11}{3},\frac{13}{3},\frac{13}{3}\right)\end{array}\right\} \therefore d(L_1,L_2)=d(P,Q)=\sqrt{\left(4-\frac{11}{3}\right)^2+\left(4-\frac{13}{3}\right)^2+\left(4-\frac{13}{3}\right)^2}$$



Q3) Sketch the level curves and the graphs of the surfaces:

a) $f(x,y)=x^2+4y^2$ $\qquad$ Dom $f=\mathbb{R}^2$

$f(x,y)=x^2+4y^2=k \quad \forall k\in\mathbb{R}^+\cup\{0\}$ are the level curves.

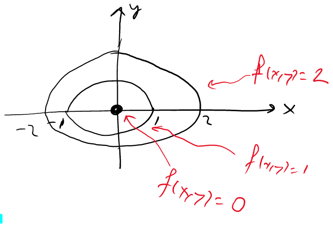
i.e. the plane $z=k$ $\cap$ the given surface

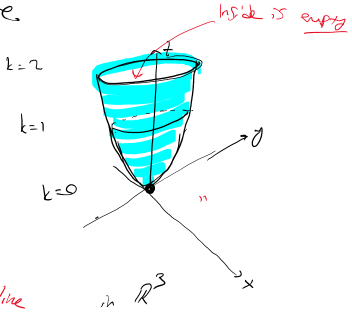
$k=0 \Rightarrow x^2+4y^2=0 \Rightarrow$ it is the pt $(0,0)$

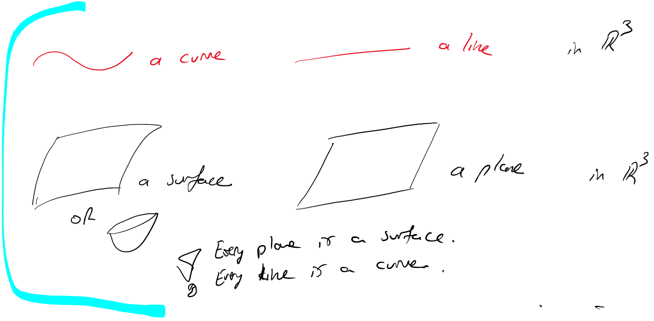
$k=1 \Rightarrow x^2+4y^2=1 \rightarrow$ ellipse

$k=2 \Rightarrow x^2+4y^2=2 \rightarrow$ ellipse

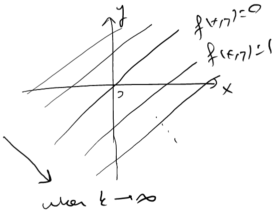
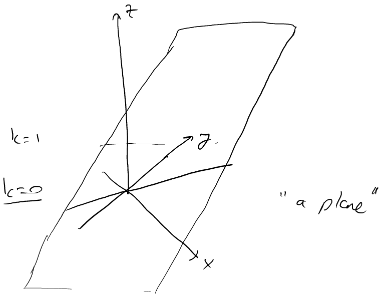
As $k$ grows, we get a larger ellipse.







∀ Every plane is a surface.
Every line is a curve.

b) $f(x,y)=2x-y$