Week 4

Friday, April 9, 2021 10:28 AM

Q1) Determine whether or not the series

$S = \ln\left(\frac{1}{2}\right) + \ln\left(\frac{3}{2}\right) + \ln\left(\frac{3}{4}\right) + \ln\left(\frac{5}{4}\right) + \ln\left(\frac{7}{6}\right) + \cdots$ converges.

$$S = \ln\left(\frac{1}{2}\right) + \ln\left(\left(\frac{2}{3}\right)^{-1}\right) + \ln\left(\frac{3}{4}\right) + \ln\left(\left(\frac{4}{5}\right)^{-1}\right) + \ln\left(\frac{7}{6}\right) + \cdots$$

$$= \ln\left(\frac{1}{2}\right) - \ln\left(\frac{2}{3}\right) + \ln\left(\frac{3}{4}\right) - \ln\left(\frac{4}{5}\right) + \cdots$$

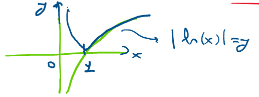
$$= \sum_{n=1}^{\infty} \underbrace{(-1)^{n+1} \ln\left(\frac{n}{n+1}\right)}_{} = a_n \qquad \text{or } a_n = (-1)^n \cdot \ln\left(1 - \frac{1}{n+1}\right)$$

i) $a_{n+1} \cdot a_n = \underbrace{(-1)^{n+1} \cdot (-1)^{n+2}}_{<0} \cdot \underbrace{\ln\left(\frac{n+1}{n+2}\right)}_{<0} \cdot \underbrace{\ln\left(\frac{n}{n+1}\right)}_{<0} < 0$

ii) $|a_{n+1}| = \left|\ln\left(\frac{n+1}{n+2}\right)\right| = \left|\ln\left(1 - \frac{1}{n+2}\right)\right| < \left|\ln\left(1 - \frac{1}{n+1}\right)\right| = |a_n|$

$1 - \frac{1}{n+2} > 1 - \frac{1}{n+1}$ & $|\ln(x)|$ is decreasing on $(0, 1]$

iii) $\lim_{n\to\infty} |a_n| = 0$ (exc!)



By A.S.T, it is convergent!

Q2) Find Maclaurin series for the given functions. State where each series converges to the function.

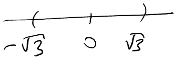
Its center is 0

a) $\frac{x}{3-x^2}$

We know that $\frac{1}{1-x} = \sum_{n=0}^{\infty} x^n$ if $|x| < 1$

$$\frac{x}{3-x^2} = \frac{x}{3\left(1 - \frac{x^2}{3}\right)} = \left(\frac{x}{3}\right) \cdot \frac{1}{1 - \frac{x^2}{3}} = \frac{x}{3} \cdot \sum_{n=0}^{\infty} \left(\frac{x^2}{3}\right)^n \quad \text{if } \left|\frac{x^2}{3}\right| < 1$$

Then, $\frac{x}{3-x^2} = \sum_{n=0}^{\infty} \frac{x^{2n+1}}{3^{n+1}}$ if $|x^2| < 3$ i.e. $|x| < \sqrt{3}$



b) $\frac{1-e^{-2x}}{x}$

Recall: $e^x = \sum_{n=0}^{\infty} \frac{x^n}{n!}$ for all x.

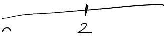
Then, $$\frac{1-e^{-2x}}{x} = \frac{1 - \sum_{n=0}^{\infty} \frac{(-2x)^n}{n!}}{x} = \frac{\cancel{1} - \cancel{1} - \sum_{n=1}^{\infty} \frac{(-2)^n \cdot x^n}{n!}}{x}$$

$$= \frac{-2x - \frac{4x^2}{2!} - \frac{8x^3}{6} + \cdots}{x} = -2 - \frac{4}{2!}x - \frac{8x^2}{6} + \cdots$$

$$= -\sum_{n=1}^{\infty} \frac{(-2)^n}{n!} x^{n-1} \quad \text{for all } x.$$

Q3) Find Taylor Series representations of the given functions where each series representation is valid?

a) $\ln(2+x)$ about $x = 2$



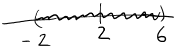
$(x-2)^n$

Recall: $\ln(1+x) = \sum_{n=0}^{\infty} (-1)^n \frac{x^{n+1}}{n+1}$ if $|x| < 1$ $(*)$

$$\ln(2+x) = \ln\left(4 + (x-2)\right) = \ln\left(4 \cdot \left(1 + \frac{x-2}{4}\right)\right) = \ln 4 + \ln\left(1 + \frac{x-2}{4}\right)$$

$$\underset{\text{by }(*)}{=} \ln(4) + \sum_{n=0}^{\infty} (-1)^n \cdot \left(\frac{x-2}{4}\right)^{n+1} \cdot \frac{1}{n+1} \quad \text{if} \quad \left|\frac{x-2}{4}\right| < 1$$

i.e. $|x-2| < 4$



b) $\frac{1}{x^2}$ about $\boxed{x=1}$

$(x-1)^n$

$$\frac{1}{x} = \frac{1}{1+(x-1)} = \frac{1}{1-(1-x)} \underset{\text{by formula of Geo. Series}}{=} \sum_{n=0}^{\infty} (1-x)^n \quad \text{if} \quad]|1-x| < 1$$

$\cancel{1} + (1-x) + (1-x)^2 + \cdots$

Take d/dx of both sides

$$\cancel{-}\frac{1}{x^2} = \sum_{n=1}^{\infty} n \cdot (1-x)^{n-1} \cdot (\cancel{-}1) \quad \text{if} \quad |1-x| < 1$$

$$\Rightarrow \frac{1}{x^2} = \sum_{n=1}^{\infty} n(1-x)^{n-1} \quad \text{if} \quad |x-1| < 1$$

or $(-1)^{n-1} \cdot n \cdot (x-1)^n$

Check the end pts!

exc!

for ex: if $x=0$, then $\sum_{n=1}^{\infty} n$ diverges. ∴ $x=0$ is not included

if $x=2$, then $\sum_{n=1}^{\infty} (-1)^{n-1} \cdot n$ diverges ∴ $x=2$ is not included

∴ $I = (0,2)$

! $f(x) = \frac{1}{x}$ ⇝ What is the power series of f around $x=0$?

it is not defined at $x=0$. ∴ It d.n. have a power series around $x=0$

Q4) Evaluate $\lim_{x \to 0} \frac{\arctan x - \sin x}{x^3 \cos x}$ $\left[\frac{0}{0}\right]$ $\underset{\text{L.H.}}{=}$ "Too long"

Recall: $\arctan x = \sum_{n=0}^{\infty} (-1)^n \cdot \frac{x^{2n+1}}{2n+1}$ if $|x| < 1$

$\sin x = \sum_{n=0}^{\infty} (-1)^n \frac{x^{2n+1}}{(2n+1)}$ $\forall x$ & $\cos x = \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n}}{(2n)!}$ $\forall x$

Then, the given limit $= \lim\limits_{x\to 0} \dfrac{\left(\cancel{x} - \frac{x^3}{3} + \frac{x^5}{5} - \cdots\right) - \left(\cancel{x} - \frac{x^3}{3!} + \frac{x^5}{5!} - \cdots\right)}{x^3 \cdot \left(1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \cdots\right)}$

$$= \lim_{x\to 0} \frac{\cancel{x^3}\left(\frac{1}{3!} - \frac{1}{3}\right) + \overset{x^2}{\cancel{x^5}}\left(\frac{1}{5!} - \frac{1}{5}\right) + \cancel{} \cdots}{\cancel{x^3}\left(1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \cdots\right)}$$

$$= \frac{\frac{1}{3!} - \frac{1}{3}}{1} = \underline{-1/6}$$

Q5) Use Maclaurin of Taylor Series to calculate the following func. value with $|\text{Error}| \leq 5.10^{-5}$ (by using A.S.E.T)

$\ln(6/5)$

we know that $\ln(1+x) = \sum\limits_{n=0}^{\infty} (-1)^n \dfrac{x^{n+1}}{n+1}$ if $|x| < 1$

Then, $\ln\left(\frac{6}{5}\right) = \ln\left(1 + \frac{1}{5}\right) = \sum\limits_{n=0}^{\infty} (-1)^n \cdot \left(\frac{1}{5}\right)^{n+1} \cdot \dfrac{1}{n+1}$

$\left|\frac{1}{5}\right| < 1$

$$\therefore \boxed{\ln\left(\frac{6}{5}\right) = \sum_{n=0}^{\infty} \underbrace{\frac{(-1)^n}{5^{n+1}\cdot(n+1)}}_{a_n}}$$

i) $a_{n+1} \cdot a_n < 0$
ii) $|a_{n+1}| < |a_n|$
iii) $\lim\limits_{n\to\infty} |a_n| = 0$

} exc!

$\therefore$ The conditions of A.S.T. are satisfied. Then,

$|E_n| = \left|\ln\left(\frac{6}{5}\right) - S_N\right| \leq |S_{N+1} - S_N| = |a_N| \leq 5.10^{-5}$

By A.S.E.T

$S_N = a_0 + a_1 + \cdots + a_{N-1}$

$S_{N+1} = a_0 + \cdots + a_{N-1} + a_N$

So, $|a_N| = \dfrac{1}{5^{N+1}\cdot(N+1)} \leq 5.10^{-5} = \dfrac{1}{20.000}$

$N=0 \implies |a_1| = 1 >$
$N=1 \implies |a_2| = \frac{1}{10} >$
$\cdots$
$N=5 \implies |a_5| = \frac{1}{18.750} >$

} Keep these terms.

$N=6 \implies |a_6| \approx \frac{1}{100000} <$ } ×

Erase all the other terms

Then, $\ln(\frac{6}{5}) \approx a_0+a_1+\cdots\cdots+a_5+\cancel{a_6}$

$\approx \underline{\underline{\sum_{n=0}^{5} a_n}}$

Q6) Find $\vec{A}\times\vec{B}$ where

$\vec{i} = \langle 1,0,0\rangle$

$\vec{j} = \langle 0,1,0\rangle$

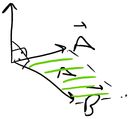
$\vec{k} = \langle 0,0,1\rangle$

a) $\vec{A} = \underset{\langle 3,-2,0\rangle}{\underset{\parallel}{3i-2j}}$, $\vec{B} = \underset{\langle 1,1,-2\rangle}{\underset{\parallel}{\vec{i}+\vec{j}-2\vec{k}}}$

$\underset{\text{vector}}{\vec{A}\times\vec{B}} = \vec{C}$ a vector perpendicular both $\vec{A}$ & $\vec{B}$

$\vec{C} = \vec{A}\times\vec{B}$

& $\|\vec{A}\times\vec{B}\| = \underline{\text{Area } A}$



$(-1)^{1+2}$

$$\vec{A}\times\vec{B} = \begin{vmatrix} i & j & k \\ 3 & -2 & 0 \\ 1 & 1 & -2 \end{vmatrix} = \langle 4-0, -(-6-0), 3-(-2)\rangle = \underline{\underline{\langle 4,6,5\rangle}}$$

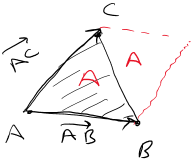
$\begin{matrix} 3 & -2 \\ 1 & 1 \end{matrix}$

b) $\vec{A} = \langle 2,1,0\rangle$, $\vec{B} = \vec{CD}$ with $C = (1,1,2)$ & $D = (2,2,1)$

$\vec{B} = \vec{CD} = D - C = (2,2,1) - (1,1,2) = \langle 1,1,-1\rangle$

Then, $\vec{A}\times\vec{B} = \begin{vmatrix} i & j & k \\ 2 & 1 & 0 \\ 1 & 1 & -1 \end{vmatrix} = \langle -1, -(-2-0), 2-1\rangle$

Q7) Find the area of triangle with vertices at $\overset{A}{(1,1,1)}$, $\overset{B}{(2,2,1)}$ & $\underset{C}{(0,1,5)}$



$2A = \|\underline{\vec{AC}\times\vec{AB}}\| = \left\| \begin{vmatrix} i & j & k \\ -1 & 0 & 4 \\ 1 & 1 & 0 \end{vmatrix} \right\| = \|\langle -4, -(-4), -1\rangle\|$

$= \|\langle -4, 4, -1\rangle\|$

$= \sqrt{4^2+4^2+(-1)^2}$

$\vec{AC} = (0,1,5) - (1,1,1) = \langle -1, 0, 4\rangle$

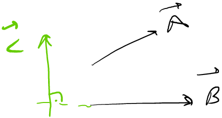
$\vec{AB} = (2,2,1) - (1,1,1) = \langle 1,1,0\rangle$

$$= \sqrt{33}$$

$$\therefore A = \underline{\sqrt{33}/2}$$

Q8) Find a vector $\vec{C}$ which is not parallel to $\vec{A} = \langle 1,2,3\rangle$ but perpendicular to $\vec{B} = \langle 1,-2,1\rangle$

$\vec{C} \not\parallel \vec{A}$ but $\vec{C} \perp \vec{B}$



$A = (1,2,4)$ a pt

$\vec{A} = \langle 1,2,4\rangle$ a vector

Choose $\vec{C} = \vec{A}\times\vec{B}$. Then $\vec{C} \perp \vec{A}$ and $\vec{C} \perp \vec{B}$.

$$\vec{C} = \vec{A}\times\vec{B} = \begin{vmatrix} i & j & k \\ 1 & 2 & 3 \\ 1 & -2 & 1 \end{vmatrix} = \langle 2 - (\overset{-6}{3\cdot -2}),\ -(1-3),\ -2-2 \rangle$$

$$= \underline{\langle +8, 2, -4\rangle}$$