Week 2

Friday, March 26, 2021 10:28 AM

Q1) Find the sum of the series $\sum_{k=1}^{\infty} \ln\left(\frac{\arctan(k+1)}{\arctan k}\right)$

$$\underline{\sum_{k=1}^{\infty} a_k} = \lim_{N\to\infty} \underbrace{\sum_{k=1}^{N} a_k}_{S_N} = \underline{\lim_{N\to\infty} S_N}$$

$$\underline{S_N} = \sum_{k=1}^{N} \ln\left(\frac{\arctan(k+1)}{\arctan(k)}\right) = \sum_{k=1}^{N} \left(\ln(\arctan(k+1)) - \ln(\arctan k)\right)$$

$$= \ln(\cancel{\arctan 2}) - \ln(\arctan 1) + \ln(\cancel{\arctan 3}) - \ln(\cancel{\arctan 2}) + \overset{\ln(\arctan 4) - \ln(\cancel{\arctan 3})}{\cdots\cdots}$$

$$+ \ln(\arctan N+1) - \ln(\cancel{\arctan N}).$$

$$= \ln(\arctan N+1) - \ln(\arctan 1)$$

Then, $\sum_{k=1}^{\infty} a_k = \lim_{N\to\infty} S_N = \lim_{N\to\infty} \left(\ln(\overset{\pi/2}{\arctan(N+1)}) - \ln(\overset{\pi/4}{\arctan 1})\right)$

$$= \ln(\pi/2) - \ln(\pi/4) = \ln\left(\frac{\pi}{2} / \frac{\pi}{4}\right) = \underline{\underline{\ln(2)}}$$

"Telescoping Series"

Q2) Let $a_n > 0$ and $\underline{a_n \leq a_{2n} + a_{2n+1}}$ for $\underline{\text{all}}$ n. Then, show that $\sum_{n=1}^{\infty} a_n$ diverges.

$a_1 \leq a_2 + a_3$

$a_2 \leq a_4 + a_5$

$a_3 \leq a_6 + a_7$

$\cdots\cdots$

$+ \quad a_N \leq a_{2N} + a_{2N+1}$

$\underline{S_N} \leq S_{2N+1} - \underline{a_1}$

Assume $\sum_{n=1}^{\infty} a_n$ is convergent.

Then, $\lim_{N\to\infty} S_N = L \; \underline{\underline{< \infty}}$

So, $\lim_{N\to\infty} S_{2N+1} = L$

As $S_N \leq S_{2N+1} - a_1$ & S_N and $S_{2N+1} - a_1$ are convergent,

$\lim_{N\to\infty} S_N \leq \lim_{N\to\infty} (S_{2N+1} - a_1)$. Then, $L \leq L - a_1$

$\underset{\text{Therefore}}{\therefore}$ $0 \geq a_1$. ↯ It is a contradiction since $a_n > 0$ for all n.

So, $\sum_{n=1}^{\infty} a_n$ diverges.

Q3) Show that if $\sum_{n=1}^{\infty} a_n$ converges, then $\sum_{n=1}^{\infty} \cos(a_n)$ is divergent.

→ for ex. $a_n = 1/n \to 0$ but $\sum_{n=1}^{\infty} \frac{1}{n}$ diverges.

$$\sum_{n=1}^{\infty} a_n \text{ converges } \overset{\not\Leftarrow}{\Longrightarrow} \lim_{n\to\infty} a_n = 0$$

~> "nth term test"

$$\lim_{n\to\infty} a_n \neq 0 \Longrightarrow \sum_{n=1}^{\infty} a_n \text{ diverges.}$$

As $\sum_{n=1}^{\infty} a_n$ converges, $\lim_{n\to\infty} a_n = 0$ by nth term test. so

$\lim_{n\to\infty} \cos(\overset{0}{\cancel{a_n}}) = 1 \neq 0$ So, by nth term test, $\sum_{n=1}^{\infty} \cos(a_n)$ is divergent

Q4) Find the sum of the following series:

a) $\sum_{n=4}^{\infty} \left(-\frac{5}{8}\right)^n = I$

⚠ $\sum_{n=1}^{\infty} r^{n-1} = \frac{1}{1-r}$ if $|r| < 1$

" $\sum_{n=0}^{\infty} r^n$ "Geometric Series"

$$I = \sum_{n=0}^{\infty} \left(-\frac{5}{8}\right)^{n+4} = \sum_{n=0}^{\infty} \left(\frac{-5}{8}\right)^n \cdot \left(\frac{-5}{8}\right)^4 = \frac{5^4}{8^4} \cdot \sum_{n=0}^{\infty} \left(\frac{-5}{8}\right)^n$$

$$\underset{\left|\frac{-5}{8}\right| < 1}{=} \frac{5^4}{8^4} \cdot \frac{1}{1-\left(-\frac{5}{8}\right)} = \frac{5^4}{8^4} \cdot \frac{8}{13} = \frac{5^4}{8^3 \cdot 13}$$

b) $\sum_{n=1}^{\infty} \sin^{2n}\theta$, $\theta \in (0, \pi/2)$

$$= \sum_{n=0}^{\infty} \sin^{2(n+1)}\theta = \sum_{n=0}^{\infty} \sin^{2n+2}\theta = \sum_{n=0}^{\infty} \sin^{2n}\theta \cdot \sin^2\theta$$

$$= \sin^2\theta \cdot \sum_{n=0}^{\infty} (\sin^2\theta)^n \underset{\text{as } |\sin^2\theta| < 1}{=} \sin^2\theta \cdot \frac{1}{1-\sin^2\theta} = \frac{\sin^2\theta}{\cos^2\theta} = \tan^2\theta$$

Q5] Test the following series for convergence

a) $\sum_{n=1}^{\infty} \frac{1}{2n^2-n} = a_n$

1st way

$$0 \le a_n = \frac{1}{2n^2-n} \le \frac{1}{2n^2-n^2} = \frac{1}{n^2} = b_n$$

as $n \le n^2$

$$\Rightarrow \sum_{n=1}^{\infty} a_n \le \underbrace{\sum_{n=1}^{\infty} \frac{1}{n^2}}$$

Conv. by Comp. test $\Longleftarrow$ Conv. by p-test as $2>1$

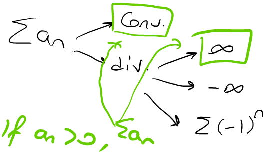
2nd way $L.C.T$ with $b_n = \frac{1}{n^2} \ge 0$

"exc"

Tests for Series

- n^{th} term Test
- Comparison test
- Limit Comp. test
- Ratio T.
- Root T.
- Integral T.
- p-test → (**)

$a_n \ge 0$



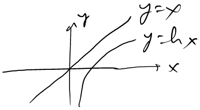
b) $\sum_{k=2}^{\infty} \frac{1}{\ln k} = a_k$

$a_k \ge 0$ and we know that $0 < \ln k < k$ $\forall k \ge 2$ by (*)

(*) Consider $h(x) = x - \ln x$ for $x>1$

$h' = 1 - \frac{1}{x} > 0$ ∴ h is increasing. So, $h(x) > \underset{1}{\underset{\parallel}{h(1)}} > 0$

$\Rightarrow x - \ln x > 0 \Rightarrow x > \ln x$ $\forall x > 1$.



So, $\frac{1}{k} < \frac{1}{\ln k} \Longrightarrow \underbrace{\sum_{k=2}^{\infty} \frac{1}{k}} < \underbrace{\sum_{k=2}^{\infty} \frac{1}{\ln k}}$

divergent by p-test (or it is a harmonic series) $\Longrightarrow$ divergent by Comparison Test

So, $\sum_{k=2}^{\infty} \frac{1}{\ln k}$ is divergent

c) $\sum_{k=1}^{\infty} \sin\left(\frac{1}{k}\right) = a_k$

$$\boxed{\lim_{x \to 0} \frac{\sin(x)}{x} = 1}$$

Consider $b_k = \frac{1}{k}$ for all $k \ge 1$. Then, a_k & $b_k \ge 0$ for all $n \ge 1$.

(as $0 < \frac{1}{k} < 1 < \frac{\pi}{2} \Longrightarrow \sin\left(\frac{1}{k}\right) > 0$)

$$\lim_{k \to \infty} \frac{a_k}{b_k} = \lim_{k \to \infty} \frac{\sin(1/k)}{1/k} \overset{[\frac{0}{0}]}{\underset{(*)}{=}} 1 \begin{smallmatrix} < \infty \\ \neq 0 \end{smallmatrix}$$

(*) $$\left[\lim_{x\to\infty} \frac{\sin(1/x)}{1/x} \overset{[\frac{0}{0}]}{\underset{L.H.}{=}} \lim_{x\to\infty} \frac{-\cancel{1/x^2}\cdot\cos\left(\frac{1}{x}\right)}{-\cancel{1/x^2}} = 1 \right]$$

By L.C.T , $\sum_{k=1}^{\infty} \sin\left(\frac{1}{k}\right) \longleftrightarrow \sum_{k=1}^{\infty} \frac{1}{k}$ are both Conv. OR div.

$\sum_{k=1}^{\infty} \frac{1}{k}$ is divergent by p-test So, $\sum_{k=1}^{\infty} \sin\left(\frac{1}{k}\right)$ diverges.

d) $\sum_{k=1}^{\infty} \left(\frac{\pi}{2} - \arctan k^2\right) = a_k$ " Convergent "

Say $b_k = \frac{1}{k^2}$, $k \geq 1$, Apply L.C.T by using $\sum_{k=1}^{\infty} \frac{1}{k^2}$

exc!

Choose $c_k = \frac{1}{k}$.

$$\lim_{k\to\infty} \frac{\pi/2 - \arctan k^2}{1/k} = \underset{?}{0}$$

$$\left(\lim_{x\to\infty} \frac{\pi/2 - \arctan x^2}{1/x} \overset{[\frac{0}{0}]}{\underset{L.H}{=}} \lim_{x\to\infty} \frac{-\frac{2x}{1+x^4}}{-1/x^2} = \lim_{x\to\infty} \frac{2x^3}{1+x^4} = \lim_{x\to\infty} \frac{2\cancel{x^3}}{\cancel{x^3}\left(\frac{1}{x^3}+x\right)} = 0 \right)$$

Then, $\underline{\sum_{k=1}^{\infty} a_k} << \underline{\sum_{k=1}^{\infty} \frac{1}{k}}$

we cannot say anything about $\sum_{k=1}^{\infty} a_n$. $\Longleftarrow$ divergent by p-test

So, $c_k = \frac{1}{k}$ is not a good choice!

e) $\sum_{n=1}^{\infty} \frac{1}{n\sqrt{n^2+3}} = a_n$

$0 \leq a_n = \frac{1}{n\sqrt{n^2+3}}$. Choose $b_n = \frac{1}{n^2} > 0$

$$\lim_{n\to\infty} \frac{\frac{1}{n\sqrt{n^2+3}}}{1/n^2} = \lim_{n\to\infty} \frac{n}{\sqrt{n^2+3}} = \lim_{n\to\infty} \frac{\cancel{n}}{\cancel{n}\sqrt{1+3/n^2}} = 1 \overset{<\infty}{\neq 0}$$

$\sum_{n=1}^{\infty} a_n \longleftrightarrow \underline{\sum_{n=1} \frac{1}{n^2}}$

Conv. by L.C.T. $\Longleftarrow$ Conv. by p-test

f) $\sum_{n=1}^{\infty} \frac{n^3}{3^n}$ "use Ratio test"

g) $\sum_{n=1}^{\infty} \boxed{\frac{n!}{n^n}} = a_n$

$a_n > 0$

$$\rho = \lim_{n\to\infty} \frac{a_{n+1}}{a_n} = \lim_{n\to\infty} \frac{(n+1)!}{(n+1)^{n+1}} \cdot \frac{n^n}{n!} = \lim_{n\to\infty} \frac{(\cancel{n+1})\cdot \cancel{n!}\cdot n^n}{(\cancel{n+1})(n+1)^n \cdot \cancel{n!}}$$

$$= \lim_{n\to\infty}\left(\frac{n}{n+1}\right)^n = \underline{\lim_{n\to\infty}\left(1-\frac{1}{n+1}\right)^n} \overset{(1^\infty)}{\underset{(*)}{=}} \underline{e^{-1}} = \frac{1}{e} < 1$$

$\boxed{\text{exc!}}$

Then, by Ratio test, $\sum_{n=1}^{\infty} a_n$ converges.

h) $\sum_{n=1}^{\infty} \boxed{\frac{1}{(n+1)^n}} = a_n$

$\underline{a_n > 0}$, $\boxed{\sigma = \lim_{n\to\infty}\sqrt[n]{a_n}} = \lim_{n\to\infty}\frac{1}{n+1} = 0 < 1$

By root test, it is conv.

∇ $a_n > 0$

0

$\sigma = \lim_{n\to\infty}\sqrt[n]{a_n}$ | $\rho = \lim_{n\to\infty}\frac{a_{n+1}}{a_n}$

$0 \le \sigma < 1 \Rightarrow \sum a_n$ conv. | $\Leftarrow 0 \le \rho < 1$

$\sigma > 1 \Rightarrow \sum a_n$ div. | $\Leftarrow \rho > 1$

$\sigma = 1 \Rightarrow$ test is not applicable | $\Leftarrow \rho = 1$

Root Test | Ratio Test

i) $\sum_{n=120}^{\infty} \frac{1}{(\ln n)^{\ln n}}$ ~

$n > \underline{\ln n}$! for $n > 120 \Rightarrow \frac{1}{(\ln n)^{\ln n}} > \frac{1}{(\ln n)^n}$

$$\Rightarrow \sum_{n=120}^{\infty} \frac{1}{(\ln n)^{\ln n}} > \underbrace{\sum_{n=120}^{\infty} \frac{1}{(\ln n)^{n}}}$$

Conv. by Root test $\left(\lim_{n\to\infty} \sqrt[n]{\frac{1}{(\ln n)^n}} = 0 < 1 \right)$

We cannot say $\Longleftarrow$
any about its convergency.

It does not work! ✗

$$(\ln n)^{\ln n} = e^{\ln\left((\ln n)^{\ln n}\right)} = e^{\ln n \cdot \ln(\ln n)} = \left(e^{\ln n}\right)^{\ln(\ln n)}$$

$$= n^{\ln(\ln n)} > n^2 \quad \text{for} \quad n > 1619$$

Here, $\ln(\ln n)$ is increasing function. If $\ln(\ln n) = 2$,

$e^2 = \ln n \Rightarrow n = e^{(e^2)} \approx 1618.\ldots$ but it should be integer.

$\therefore \quad n^{\ln(\ln n)} > n^2$ for $n > 1618$

OR $n > [\![e^{(e^2)}]\!] + 1$

In fact, $0 < a_n = \frac{1}{(\ln n)^{\ln n}} = \frac{1}{n^{\ln(\ln n)}} < \frac{1}{n^2}$ for $n \geq 2021$

$$\Rightarrow \sum_{n=2021}^{\infty} a_n < \underbrace{\sum_{n=2021}^{\infty} \frac{1}{n^2}}$$

Conv. by Comp. test. $\Longleftarrow$ Conv. by p-test as $2 > 1$

What about $\sum_{n=120}^{\infty} a_n$?

$$\sum_{n=120}^{\infty} a_n = \underbrace{a_{120} + a_{121} + \cdots + a_{2020}}_{\in \mathbb{R}} + \underbrace{\sum_{n=2021}^{\infty} a_n}_{\text{it is convergent}} \quad \text{is convergent.}$$