Week 1

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$\boxed{\text{5 pts (Bonus)}}$

Check the website of Math 120 & ODTUClass page

For the questions you want to ask, there will be Online office hours & Helproom.

Q1) Find the limit of the sequence $\{a_n\}$, where $\boxed{\lim_{n\to\infty} a_n}$

a) $a_n = \dfrac{\cos(4n)}{n+2}$

$$0 \le \left|\frac{\cos(4n)}{n+2}\right| \le \frac{1}{n+2} \quad \text{for } n \geq 1$$

as $n \to \infty$

0

By Squeeze thm, $\lim_{n\to\infty} |a_n| = 0$. Then, $\lim_{n\to\infty} a_n = 0$.

$$\left[-\frac{1}{n+2} \le a_n \le \frac{1}{n+2} \quad \dots \right]$$

b) $a_n = n - \sqrt{n^2-4n}$

$$\lim_{n\to\infty} \left(n-\sqrt{n^2-4n}\right) \overset{[\infty-\infty]}{=} \lim_{n\to\infty} \frac{(n-\sqrt{n^2-4n})\cdot(n+\sqrt{n^2-4n})}{(n+\sqrt{n^2-4n})}$$

$$= \lim_{n\to\infty} \frac{n^2-(n^2-4n)}{n+\sqrt{n^2-4n}} = \lim_{n\to\infty} \frac{4n}{n+\sqrt{n^2(1-4/n)}}$$

$|n|$ & $n>0$

$$= \lim_{n\to\infty} \frac{\cancel{n}\cdot 4}{\cancel{n}(1+\sqrt{1-\cancel{4/n}}\xrightarrow{}0)} = \frac{4}{1+1} = \underline{\underline{2}}$$

c) $a_n = \left(\dfrac{n+2}{n+1}\right)^n$

$$\lim_{n\to\infty}\left(\frac{n+2}{n+1}\right)^n \overset{[1^\infty]}{=}$$

Observe that $f(n)=a_n$ comes from $f(x)=\left(\frac{x+2}{x+1}\right)^x$

(then $f(n)=a_n$). ∇ θ $f:\mathbb{R}\longrightarrow\mathbb{R}$

$$\lim_{x\to\infty}\left(\frac{x+2}{x+1}\right)^x \overset{[1^\infty]}{=} \lim_{x\to\infty} e^{\ln\left(\left(\frac{x+2}{x+1}\right)^x\right)} = \lim_{x\to\infty} e^{x\cdot\ln\left(\frac{x+2}{x+1}\right)}$$

$$= e^{\lim_{x\to\infty} x\cdot\ln\left(\frac{x+2}{x+1}\right)} = e^{\boxed{\lim_{x\to\infty}\frac{\ln\left(\frac{x+2}{x+1}\right)}{1/x}}} \ (*)$$

as e^x is cont.

$$(*) \overset{[\frac{0}{0}]}{\underset{L.H}{=}} \lim_{x\to\infty}\frac{\frac{x+1}{x+2}\cdot\left(-\frac{1}{(x+1)^2}\right)}{-1/x^2} = \lim_{x\to\infty}\frac{x^2}{(x+1)(x+2)} = 1$$

$$\therefore \lim_{x\to\infty} f(x) = e^1 \Longrightarrow \lim_{n\to\infty} a_n = e$$

Let ∇ θ $f:\mathbb{R}\to\mathbb{R}$ & $f(n)=a_n$. $\lim_{x\to\infty} f(x) = L \Longrightarrow \lim_{n\to\infty} f(n) = \lim_{n\to\infty} a_n = L$

$\not\Longleftarrow$ Counter ex. in Q3.

d) $a_n = \dfrac{n!}{n^n}$

$$\frac{1}{n} = 1\cdot\ldots\cdot1\cdot\frac{1}{n} \geq a_n = \frac{n!}{n^n} = \frac{\cancel{n}\cdot(n-1)\cdot\ldots\ldots 2\cdot 1}{\underbrace{\cancel{n}\cdot n\cdot\ldots\ldots n\cdot n}_{n\text{-many}}} = \underset{1\geq}{\frac{n-1}{n}}\cdot\underset{1\geq}{\frac{n-2}{n}}\cdots\underset{1\geq}{\frac{2}{n}}\cdot\frac{1}{n} \geq 0$$

as $n\to\infty$

0

By Squeeze thm, $\lim_{n\to\infty} a_n = 0$.

Q2) Let (a_n) be the seqn. given as follows: $n\geq1$

$$a_n = \begin{cases} \dfrac{2}{2n^3+n+1} & \text{if } n \text{ is even} \\[2ex] \dfrac{-3}{n^2+3n+5} & \text{if } n \text{ is odd} \end{cases}$$

If exists, find $\lim_{n\to\infty} a_n$

$$0 < \frac{2}{\underbrace{2n^3+n+1}_{>0}} < \frac{2}{n} \quad \text{if } n \text{ is even}$$

$$0 > \frac{-3}{\underbrace{n^2+3n+5}_{>0}} > \frac{-3}{3n} = -\frac{1}{n} \quad \text{if } n \text{ is odd}$$

So,

$$-\frac{1}{n} \le a_n \le \frac{2}{n} \quad \text{for } n \ge 1$$

as $n\to\infty$ → 0

By Squeeze thm, $\lim_{n\to\infty} a_n = 0$ (∴ It exists)

2nd way

If n is even, $\lim_{n\to\infty} \frac{2}{2n^3+n+1} = 0$.

If n is odd, $\lim_{n\to\infty} \frac{-3}{n^2+3n+5} = 0$

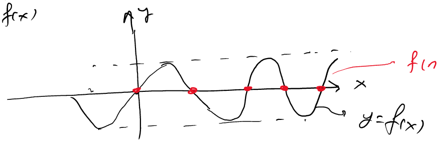
& both sequences approach to the same number 0.

∴ $\lim_{n\to\infty} a_n = 0$.

Q3) Find the limit of the seqn. $\{a_n\}$, where $a_n = \sin(n\pi)$.

$\lim_{n\to\infty} a_n = \lim_{n\to\infty} \sin(n\pi) = 0$ as $\sin(n\pi) = 0$ for $n \in \mathbb{N}$

$\lim_{x\to\infty} \underset{f(x)}{\sin(x\pi)} =$ d.n.e. but $\lim_{n\to\infty} f(n) = a_n = 0$



Q4) Consider the seqn (a_n) defined recursively as follows:

$a_1 = 1$, and $a_n = 1 + \frac{a_{n-1}}{2}$ for all $n \ge 2$.

a) Show that $a_n < 2$ for all n.

Initial Step For $n=1$, $a_1 = 1 < 2$ ✓

Inductive Step Assume $a_k < 2$. Then, $a_{k+1} = 1 + \frac{a_k}{2} < 1 + \frac{2}{2} = 2$ ✓
by assump.

∴ $a_n < 2$ for all n. (by induction) ((a_n) is bounded above.)

b) Show that $a_n < a_{n+1}$ ∀n.

Initial Step For $n=1$, $a_2 = 1+\frac{1}{2} > a_1 = 1$ ✓

Inductive Step Assume $\underline{a_{k+1} > a_k}$. Then,

$$a_{k+2} \underset{\text{by def'n}}{=} 1 + \frac{a_{k+1}}{2} \underset{\text{by assump.}}{>} 1 + \frac{a_k}{2} = a_{k+1}$$

$\therefore$ By induction, $\underline{a_{n+1} > a_n}$ $\forall n \geq 1$. ((a_n) is increasing seqn.)

c) If exists, find the limit $\lim_{n\to\infty} a_n$.

Since a_n is increasing seqn. & bounded above, by Monotone Convergence thm, it's limit exists. Say $\lim_{n\to\infty} a_n = L$. Then $\lim_{n\to\infty} a_{n+1} = L$

We have $a_{n+1} = 1 + \frac{a_n}{2}$. Then

$$\underset{L}{\underset{\parallel}{\lim_{n\to\infty} a_{n+1}}} = \underset{1+\frac{L}{2}}{\underset{\parallel}{\lim_{n\to\infty}\left(1+\frac{\overset{L}{a_n}}{2}\right)}} \implies L = 1 + \frac{L}{2} \implies 2L = 2 + L \implies \boxed{L=2}$$

Q5) Let (a_n) be the seqn. defined recursively as follows:

$a_1 = 1$, $a_2 = 2$ & $a_n = a_{n-1} + a_{n-2}$ for all $n \geq 3$. → "Fibonacci Seqn."

You are given that the seqn. $\left(\frac{a_{n+1}}{a_n}\right)$ has a positive limit L. Find the limit $\lim_{n\to\infty}\left(\frac{a_{n+1}}{a_n}\right)$. ⤳ "Golden Ratio"

$$a_n = a_{n-1} + a_{n-2} \underset{\substack{\text{increase the indexes} \\ \text{by 1}}}{\implies} a_{n+1} = a_n + a_{n-1}$$

As $a_n \neq 0$, $\frac{a_{n+1}}{a_n} = \frac{a_n}{a_n} + \frac{a_{n-1}}{a_n} = 1 + \frac{a_{n-1}}{a_n}$

Take the limit of both sides:

$$L = \lim_{n\to\infty}\left(\frac{a_{n+1}}{a_n}\right) = \lim_{n\to\infty}\left(1+\overset{1/L}{\frac{a_{n-1}}{a_n}}\right) \underset{(*)}{=} 1+\frac{1}{L}$$

$$(*)\left[\lim_{n\to\infty}\frac{a_{n-1}}{a_n} \underset{\substack{m=n-1\\ \Downarrow\\ n=m+1}}{=} \lim_{m\to\infty}\frac{a_m}{a_{m+1}} = \lim_{m\to\infty}\frac{1}{\frac{a_{m+1}}{a_m}} \underset{L\neq 0}{=} \frac{1}{L}\right]$$

So, $L = 1+\frac{1}{L} \Longrightarrow L^2-L-1=0 \Longrightarrow L=\frac{1\mp\sqrt{5}}{2}$ but $L>0$

$$\therefore \boxed{L=\frac{1+\sqrt{5}}{2}}$$

① $a_n=\frac{(-1)^n}{n}$ is neither increasing nor decreasing but it is convergent

Q: T/F

T • $\{a_n\}$ is an increasing seqn. $\Longrightarrow$ arctan(a_n) is convergent

a_n is increasing. So, $a_{n+1} > a_n$ for all n. Since arctan x is an increasing func., arctan$(a_{n+1}) >$ arctan(a_n). So, arctan(a_n) is increasing seqn. and also it is bounded above by $\frac{\pi}{2}$. $\therefore$ It is convergent. "True"

F • $0 \leq a_n < b_n$ and $\{a_n\}$ is divergent. Then $\{b_n\}$ is so.

Chose $a_n=(-1)^n+1$ & $b_n=\frac{1}{n}+120$

$0\leq a_n < b_n$ and a_n is divergent. BUT, $\lim_{n\to\infty} b_n = 120$ (It is conv.).

"False"

⓪ $a_n=(-1)^n$ is bounded seqn but it is divergent.